Chapter 4

Time-Refraction in Classical and Quantum Optics

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4.1 Introduction

Temporal effects in optics, and space-time symmetries, have been considered for a long time [1, 2]. More recently, the concepts of time-refraction and time-reflection were explicitly defined, using a classical approach, in [3] and more recently, in [4]. A quantum formulation of these concepts can also be found, see [5, 6]. This concerns propagation in non-stationary optical media, where boundary effects can usually be ignored. In parallel, a closely related problem was considered, where boundaries are a necessary ingredient, concerning the emission of photons from vacuum due to the existence of moving mirrors. This was first studied by Moore [7] and others [8], and became known as the dynamical Casimir effect (recently reviewed by Dodonov [9], see Figure 4.1). Experimental analogues of quantum cosmology using time-varying optical media have also been proposed [10].

An important property of time-refraction is that it is a first order effect, which concerns every photon propagating in the medium, in the same way as the usual space-refraction. For that reason, it can easily be observed at the macroscopic classical level, using a photon beam of arbitrary intensity. Several manifestations of time-refraction have indeed been identified. In the classical regime they are mainly two: frequency shift and time-reflection. Light amplification by successive time-refraction events is also possible. We should also add the emission of photon-pairs, which is only relevant in the quantum regime.

In contrast with classical effects, such as frequency shift and refraction, the quantum properties of time-refraction in a time-varying medium only concern a small number of photons, due to a very low emission probability. The same occurs for the dynamical Casimir effect associated with a moving boundary, which is a closely connected process. For that reason, the quantum manifestations of time-refraction are much more difficult to find. We should however mention that observation of the dynamical Casimir effect has be claimed in superconducting circuits [11].

Here we review time-refraction in classical and quantum optics and describe its main physical features. We start with a description of the main properties of time-refraction in classical optics, which include frequency shits, *time-reflection* and light amplification. This topic has recently taken considerable relevance in the context of metamaterials [12–17] (see also the review article [18]). In magnetized media, the occurrence of temporal boundaries can also induce a Faraday rotation [19].

The cases of i) a single temporal boundary, ii) a *temporal beam-splitter* made of two consecutive temporal boundaries, iii) and generalization to *time-crystals* made of a succession of temporal beam-splitters, are considered. A quantum-optical description of these processes is also possible and reveals the possible creation of photon pairs in quantum vacuum. This photon pair creation is indeed very similar to that of the dynamical Casimir process, and in some sense, time-refraction generalizes the concept of dynamical Casimir, because no optical boundary conditions are needed. On the other hand, time-refraction is a first-order optical process, which exists in both classical and quantum regimes, affects all the photons in the medium and is independent of boundaries. In contrast, the dynamical Casimir is a second order quantum effect which depends on the existence oscillating boundaries.

Finally, we consider the strong field regime, associated with intense laser pulses, where temporal optical effects can be related to electron-positron pair creation. This new regime is the so-called laser QED. In order to illustrate it, we focus on the temporal Klein model, which is a temporal version of the well-known Klein paradox. This model describes a kind of time-refraction for the Dirac field, and explains the creation of particle pairs in a non-stationary vacuum. It can also be generalised to arbitrary non-stationary fields, and leads to a quantum kinetic equation describing the temporal evolution of the electron and positron populations. This kinetic equation is ultimately related with vacuum breakdown.

4.2 Time-Refraction and Time-Reflection

We start with a short discussion of the basic space-time symmetries. The usual (space) refraction is associated with space symmetry breaking, where photon energy (but not photon momentum) is conserved. This occurs in the presence of a boundary between two different optical media, with refractive indices n_0 and n_1, for instance defined at a given (x, y)-plane, described by $z = 0$. Such a boundary destroys translation invariance and introduces a change of momentum for light crossing the boundary. Photons at a given frequency, will receive from the existing boundary the exact amount of momentum necessary to satisfy the dispersion relation on both media. This is described by the famous Snell's law, $n_0 \sin\theta_0 = n_1 \sin\theta_1$, where $\theta_{0,1}$ are the angles of incidence and refraction.

Similarly, we can replace the space boundary by a temporal boundary, which corresponds to a sudden change of the refractive index at some instant of time $t = 0$, from n_0 to n_1. Here, space boundaries are ignored, and the medium can be assumed as infinite. This temporal symmetry breaking can be described in the (x, t)-place, as illustrated in Figure 4.2. By analogy with the usual (space) refraction, we can call it *time-refraction*. In this case, photons crossing the temporal boundary conserve momentum, but change their frequency, in order to to satisfy the dispersion relations, before and after the transition.

Figure 4.1: *Victor Dodonov (on the left) with the author, during his visit to Lisbon in December 2010. The central figure is the bronze statue of a lottery ticket man, typical character on the streets of Lisbon in the early 20th century.*

This leads to a new relation, called the *temporal Snell's law*, and given by

$$\omega_0 \tan \alpha_0 = \omega_1 \tan \alpha_1 \,, \tag{4.1}$$

where $\alpha_{0,1} = \tan^{-1}(n_{0,1})$ are the angles of temporal incidence and refraction, defined in the (x,t) plane. These two angles define the group velocity of the photons, before and after the temporal transition. The two different Snell's laws can be seen as particular cases of a *generalised Snell's law*, that takes the form [25]

$$n_0 \sin \theta_0 = \left(\frac{\omega_1}{\omega_0} \right) n_1 \sin \theta_1 \tag{4.2}$$

The first reported observation of a frequency shift associated with the temporal change of the refractive index was probably made by Yablonovich in 1974 [20], when a CO_2 laser pulse was used to ionize Nitrogen and Helium gas jets. In these experiments, the observed spectrum of the laser radiation was considerably blue-shifted after ionization. For that reason, the process was initially named *self-blue shift*. It is now currently observed in short laser pulse interactions with gas-jets (see [21–23]). Sometimes, this elementary (and purely temporal) process is mixed with other spatiotemporal effects, and leads to optical processes with different names, such as, self-phase and cross-phase modulation, photon acceleration and pulse compression [24, 25]. Recent experiments on time-refraction can be found in [26, 27].

Notice that, in order to satisfy the field continuity conditions (to be discussed next), we need to include a reflected signal. But, because reflection is not allowed in time (unfortunately, we cannot travel back in time) this is a reflected signal in space. Therefore, time-refraction also necessarily implies partial reflection in space. This means that, at the classical level, time-refraction implies the existence of two distinct phenomena: frequency shift and (space) reflection. It is important to stress again that time-refraction is (just like the usual space refraction) a basic first order process with hundred-percent efficiency, affecting all photons propagating through a temporal boundary. If the photons belong to a laser pulse, the entire pulse will change frequency, according to the general law (4.2).

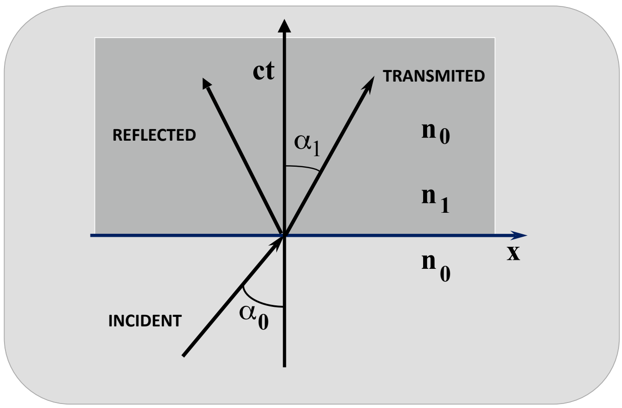

Figure 4.2: *Time-refraction in the (x, t) plane: the optical medium suffers a sudden change of its refractive index, at $t = 0$, from n_0 to n_1. Incident, reflected and transmitted rays are represented.*

In such a simple temporal configuration, the continuity equations for the electromagnetic field can easily be established. They essentially imply the continuity of the displacement and magnetic fields, $\mathbf{D}(t)$ and $\mathbf{B}(t)$, valid for all times, including the discontinuity $t = 0$ [5]. For a light beam propagating in the medium with initial electric field amplitude $\mathbf{E}_0$, we can then derive the amplitudes of the temporal transmitted and reflected fields, $\mathbf{E}_1$, and $\mathbf{E'}_1$. They are determined by what can be called the *temporal Fresnel's laws*, which take the form

$$R \equiv \frac{E'_1}{E_0} = \frac{\alpha}{2}(\alpha - 1) \,, \quad T \equiv \frac{E_1}{E_0} = \frac{\alpha}{2}(\alpha + 1) \,, \tag{4.3}$$

where $\alpha = n_0/n_1$ is the ratio between the initial and the final refractive index of the optical medium. Notice that $R + T = \alpha^2$, which indicates that the electromagnetic wave energy is not conserved during the process. An exchange of energy between the field and the medium should necessarily take place, but is not explicitly described by the field boundary conditions, and is usually ignored.

We can slightly improve the above discussion by assuming that, for times $t < 0$ prior to time-refraction, we have two light beams of the same frequency and polarization but propagating in opposite directions. Defining $\mathbf{E}_0$ as the field amplitude of beam propagating with wavevector $\mathbf{k}$, and $\mathbf{E}'_0$ as the field amplitude of beam propagating in the opposite direction, $-\mathbf{k}$, the above reflection and transmission coefficients are replaced by [28]

$$R \equiv \frac{E_1'}{E_0} = \frac{\alpha}{2}\left[(\alpha-1)+(\alpha+1)\frac{E_0'}{E_0}\right] , \quad T \equiv \frac{E_1}{E_0} = \frac{\alpha}{2}\left[(\alpha+1)+(\alpha-1)\frac{E_0'}{E_0}\right] , \tag{4.4}$$

In this case, qualitatively new effects appear. In particular, we have the possible occurrence of *total reflection*, $T = 0$, where no transmitted signal is observed after $t = 0$. This can take place when the two initial field amplitudes satisfy the relation $E_0'/E_0 = (\alpha+1)/(1-\alpha)$. In contrast, we can observe *total transmission*, $R = 0$, when the initial field amplitudes satisfy the relation $E_0'/E_0 = (1-\alpha)/(1+\alpha)$. This total transmission condition is the temporal analogue of the well-known Brewster's angle and can be called the *temporal Brewster's angle*, defined in the spacetime plan (x, t). This concept was recently explored in [29].

The same kind of discussion can be transposed from classical to quantum optics, where the field amplitudes are replaced by field operators. Similar continuity relations can be established for quantum field operators. In this case, description of photon emission from vacuum becomes possible. Let us consider a Fock state of n photons with frequency ω_1 propagating in the forward direction, with no photons in the backward direction, as defined by the state vector $|n, 0\rangle$. It can than be shown that, after a time-refraction event at $t = 0$, we observe states with both forward and backward photons, defined by the new state vector $|n+s, s\rangle$, where s is the number of photon-pairs created by the temporal discontinuity. This photon-pair creation process can be described by [5]

$$|n, 0\rangle_1 = \sum_{s=0}^{\infty} b_s(n)\, |n+s, s\rangle_2 \ , \tag{4.5}$$

where the probability for the occurrence of different photons states $P(n+s, s) = |b_s(n)|^2$ is determined by

$$P(n+s, s) \equiv |b_s(n)|^2 = \frac{(n+s)!}{n!s!}\left(\frac{B^s}{A^{n+s+1}}\right)^2 , \tag{4.6}$$

where the two coefficients A and B are defined by

$$A = \frac{(1+\alpha)}{2\sqrt{\alpha}} , \quad B = \frac{(1-\alpha)}{2\sqrt{\alpha}} . \tag{4.7}$$

Notice that these coefficients satisfy the hyperbolic relation $A^2 - B^2 = 1$, characteristic of a bosonic field. Obviously, the $(n+s)$ photons propagating in the forward direction, with wavevector $\mathbf{k}$, correspond to the transmitted signal, while the s photons propagating in the backward direction $-\mathbf{k}$ correspond to the reflected signal. Given the final value n_1 of the refractive index, both signals propagate in the new medium with a shifted frequency, determined by $\omega_1 = |k|c/n_1$.

4.3 Temporal Beam-Splitters

As a natural extension of the basic time-refraction process described above, we can consider the temporal beam-splitter, which is the temporal analogue of the well-known optical beam-splitter. It is defined in the following way. We assume that, at $t = 0$ the refractive index suddenly changes from its initial value n_0 to a different value n_1, and then, after a time interval τ, the refractive index returns to its initial value n_0. This time interval is the analogue of the width of the optical beam-splitters. Of course, the main difference is that there are three independent directions in space, with only a single direction in time. Using the two consecutive continuity relations, as above, we can derive the time-transmitted and time-reflected coefficients in the form [28]

$$R_2 \equiv \frac{E_2'}{E_0} = \frac{i}{2\alpha}\left(1 - \alpha^2\right)\sin(\omega_1\tau)\exp(-i\omega_0\tau)\,, \tag{4.8}$$

and

$$T_2 \equiv \frac{E_2}{E_0} = \left[\cos(\omega_1\tau) - \frac{i}{2\alpha}\left(1 + \alpha^2\right)\sin(\omega_1\tau)\right]\exp(+i\omega_0\tau)\,, \tag{4.9}$$

where E_2 and E_2' are the final transmitted and reflected field amplitudes, after the two consecutive temporal discontinuities. For simplicity, we have assumed that no reflected signal is initially present, $E_0' = 0$, as in eq. (4.3). But the general case with $E_0' \neq 0$ can be equally defined. We clearly see that these coefficients oscillate as a function of the temporal width τ. This is obviously a result of interferences created by the two temporal surfaces. As before, the maximum values of these coefficients are defined by the ratio between the two refractive indices, α. Notice that, when $\alpha \to 0$, or $n_0 \simeq 0$, the maximum reflected and transmitted fields can grow indefinitely, showing an increase of the total energy of the radiation field.

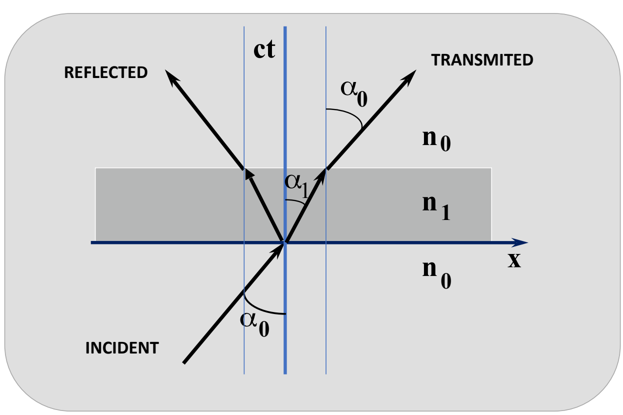

Figure 4.3: *Temporal beam-splitter, made of two successive time-refraction events: First, at $t = 0$ the refractive index suddenly changes, from n_0 to n_1, and then at $t = \tau > 0$, the refractive index returns to its initial value n_0. Incident, reflected and transmitted rays across the temporal slab of duration τ are represented.*

This concept of temporal beam-splitter was recently revisited, for short pulse propagation in metamaterials [13], where the above analytical results were confirmed by numerical simulations, and more complicated cases were also numerically solved. It should however be noticed that analytical solutions can also be derived for pulses with arbitrary temporal shape. This analytical approach is based on the analogy with wave propagation in inhomogeneous optical media, such as, in stratified media. It is actually quite easy to derive time-transmission and time-reflection coefficients for propagation in a generic non-stationary medium, where the refractive index is an arbitrary function of time, $n(t)$. This leads to the following expressions for the transmitted and reflected electric field amplitudes [30]

$$\frac{dE}{dt} = -\frac{1}{2n}\frac{dn}{dt}\left[3E + E' \exp(+i\varphi(t'))\right] , \tag{4.10}$$

and

$$\frac{dE'}{dt} = -\frac{1}{2n}\frac{dn}{dt}\left[3E' + E \exp(-i\varphi(t))\right] , \tag{4.11}$$

where the phase $\varphi(t)$ is defined by

$$\varphi(t) = 2\int_0^t \omega(t')dt' , \quad \omega(t) = \frac{kc}{n(t)} . \tag{4.12}$$

Here, the time-dependent wave frequency $\omega(t)$ is determined by the instantaneous photon dispersion relation, assumed as locally valid, as indicated. But more intricate relations with nonlinear and nonlocal dispersion terms can also be assumed. It is particularly useful to consider the case of a slowly varying medium, where the incident wave is always dominant, and we have $|E'| \ll |E|$. In this case, the time-reflection and time-transmition coefficients are then given by

$$R(t) \equiv \frac{E'(t)}{E(0)} = -\int_0^t \frac{1}{2n}\frac{dn}{dt'} \exp\left[-i\varphi(t')\right] dt' , \tag{4.13}$$

and

$$T(t) \equiv \frac{E(t)}{E(0)} = \exp\left[-3\int_0^t \frac{1}{2n}\frac{dn}{dt'}dt'\right] . \tag{4.14}$$

It is interesting to notice that these equations are the exact temporal analogues of the reflection and transmission coefficients in stationary inhomogeneous media, where the refractive index varies in space and not in time, as discussed in several books (see, for instance, [31, 32]).

4.4 Time-Crystals

The time-refraction process is also intimately related with the concept of *time crystal*. This concept was initially imagined as a result of spontaneous symmetry breaking, leading the system to the lowest energy state [33]. In this final state, the system would spontaneously oscillate in time, in analogy with the periodic space arrangements of the ordinary crystals. However, it was soon realized that time crystalization cannot spontaneously occur, because it needs to be driven by an external energy source [34], therefore not in isolated systems. Subsequently, many different configurations of driven time crystals have been proposed in different media (see for a review [35]), such as in ultra-cold atomic matter [36].

The simplest way to obtain a time crystal is to built a temporal-cavity, which means to superpose a large number $N \gg 1$, of identical and equally spaced temporal beam-splitters, distant from each other by a duration ΔT, as represented in Figure 4.4. In alternative, we could assume periodic modulations of a continuously varying medium, where the influence of space boundaries can once more be forgotten. For this purpose, we consider an optical medium where the refractive index oscillates at a given frequency $\omega_c = 1/\Delta T$, as described by

$$n(t) = n_0 \left[1 + \epsilon_c \cos(\omega_c t)\right] , \tag{4.15}$$

where ϵ_c is the amplitude of the oscillation. Applying the quantum formulation for non-stationary media [30], we can state that the number of photon pairs emitted in a given wavevector mode $\mathbf{k}$ is given by

$$\langle N_{\mathbf{k}}(t) \rangle = \sinh^2 \left(\int^t |\eta(t')| \, dt' \right) , \tag{4.16}$$

where the quantity $\eta(t)$ can be written in the form

$$\eta(t) = \frac{\epsilon_c \omega_c}{2} \sin(\omega_c t) \sum_\nu J_\nu(\epsilon_c) \exp\left[-i(2\omega + \nu\omega_c)t)\right] , \tag{4.17}$$

where $J_\nu(\epsilon_c)$ are Bessel functions. The main contribution to the emission of photon pairs is clearly given by the constant part of the function $\eta(t)$, because the integral over the oscillating part over a long time interval in eq. (4.16) will be averaged to zero. This constant part is defined by the condition

$$\omega = \frac{1}{2}(\nu \pm 1)\omega_c \, . \tag{4.18}$$

This formula defines what can be seen as a *temporal Bragg diffraction*, which is the condition to attain a maximum value of backscattered (or time reflected) light. But, due to the need to satisfy total momentum conservation, it also corresponds to a maximum of time transmitted light. These peaks of the back and forward scattered signals occur when the frequency ratio (ω/ω_c) takes the value of $1/2$, for $(\nu = 0)$, which is the well-known dynamical Casimir condition, and more generally of $(\nu \pm 1)/2$, for ν integer. This shows that a time crystal is intimately related with the dynamical Casimir effect, as it is associated with a temporal oscillation of the optical path in the medium. Still, we are very far from the usual model of the dynamical Casimir effect, based on an oscillating mirror.

4.5 Superluminal fronts

Until now, we have only considered the case of temporal changes in unbounded media. But, if we assume the simultaneous effects of space and time variations, we can have access to new kinds of phenomena, such as photon acceleration (mainly known in plasmas) and

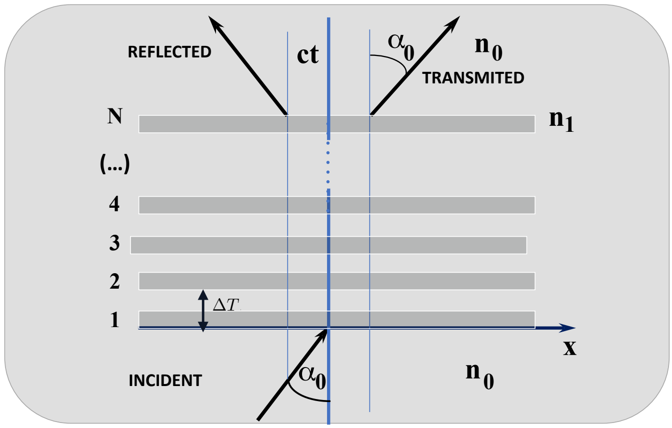

Figure 4.4: *Time-crystal, made of $N \gg 1$ successive temporal beam-splitters events with the same duration τ, and the same refractive index n_1, on a background of refractive index n_0, separated by a time interval ΔT.*

self-phase modulation (known in nonlinear optics), which have been mainly discussed in the frame of classical physics. There are some particular cases, however, where simple quantum descriptions have been proposed. This is the case of perturbations moving with superluminal velocity, such as superluminal ionization fronts [37] (see also the related work [43]), that we will now briefly discuss.

Relativistic ionization fronts usually move with velocities below the speed of light, and have been studied theoretical and experimentally for very long time [44]. They can be described by sharp boundary layers between a neutral and an ionized gas. The basic principle of ionization fronts is that of an optical boundary with relativistic velocity, produced by an intense laser pulse, in the absence of motion of atoms and charged particles. This relativistic boundary separates the plasma region from the neutral gas region. On one side of the front we have free electrons and ions, while on the other side we only have neutral atoms. The motion of this boundary is similar to that of a fire front, which can move very fast in a forest, even if the trees are immobile and attached to the ground by nature. The behaviour of a probe beam with frequency ω_i crossing this boundary can be described by a refractive index $n(\mathbf{r}, t)$ of the form

$$n(\mathbf{r}, t) = \left[1 - \frac{\omega_p^2(\mathbf{r}, t)}{\omega_i^2} \right]^{1/2} . \tag{4.19}$$

The plasma frequency ω_p is determined by the free electron density, and varies abruptly across the front. It can be described by an expression of the form

$$\omega_p^2(\mathbf{r}, t) = \frac{1}{2}\omega_{p0}^2 \left\{ 1 + tanh\left[k_f q(x, t) \right] \right\} , \quad q(x, t) = k_f (x - ut) , \tag{4.20}$$

where we have assumed a front velocity moving along an arbitrary x-axis, with velocity $\mathbf{u} = u\mathbf{e}_x$. The quantity k_f defines the width of the front, and ω_{p0} is a constant determining the maximum value of the plasma frequency attained behind the front.

Superluminal fronts, with $u > c$, although not easy to create, are possible because these are optical boundaries, not related with any actual particle motion. There are several ways to produced these superluminal fronts in the laboratory, in particular, using the *flying focus* concept [45, 46]. This concept has recently been explored and is mainly based on the use of short laser pulses with a broad spectrum, combined with chromatic optical components. For instance, the use of a chromatic lens with a focal distance that depends on the frequency. When a short pulse is focused by such lenses, the different parts of the spectrum will focus at different positions and, if sufficiently intense, they will ionize the neutral gas at different instants and different locations, eventually creating an ionization front that moves with an arbitrarily velocity. Such optical arrangements have been explored by different groups in simulations and experiments, and are not in contradiction with special relativity.

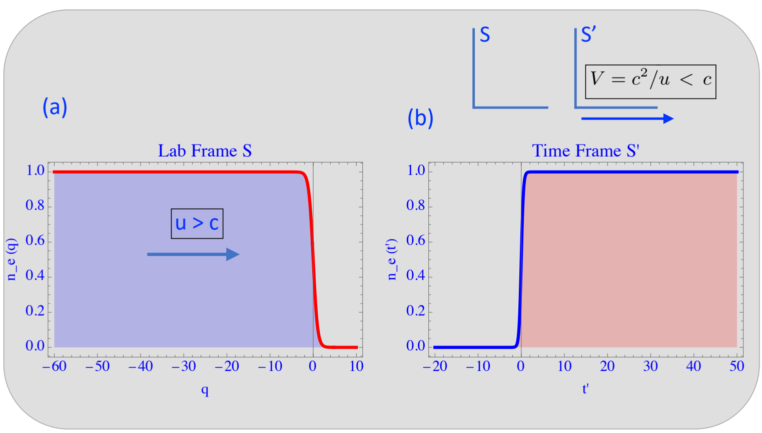

Figure 4.5: *Superluminal front: Normalized plasma density $n_e(x)/n_{e0}$: (a) in the lab frame S, and (b) in the time frame S', moving with velocity $V = c^2/u < c$ with respect to S.*

Apart from a pioneering work based on geometric optics [37], superluminal fronts have also been theoretically described in the frame of quantum optics, several years before the appearance of these recent flying focus devices, see [30, 47]. Such theoretical models have shown that superluminal fronts can be reduced to a purely temporal process, identical to time-refraction, if we use an adequate reference frame. Making a Lorentz transformation to a moving frame S', such that it moves with respect the laboratory frame S with an appropriate velocity V, and considering the well-known velocity transformation formula

[48], we have

$$u = \frac{u' + V}{1 + Vu'/c^2} \,. \tag{4.21}$$

We realize that, for $u > c$, the front velocity in the new frame will diverge, $u' \to \infty$, if we choose $V = c^2/u < c$. Noting that the plasma frequency ω_p is a relativistic invariant, we can easily conclude that, in the coordinates $(\mathbf{r}', t')$ of the moving frame, this quantity only depends on time, and will be described by

$$\omega_p^2(t') = \frac{\omega_{p0}^2}{2}\left[1 + tanh(\nu'_f t')\right] \,, \quad \nu'_f = k_f \gamma (V - u) \,, \tag{4.22}$$

where $\gamma = 1/\sqrt{1-\beta^2}$, with $\beta = V/c$. Notice that ν'_f is negative, as illustrated in Figure 4.5. This means that the superluminal front is reduced to a time-refraction event at $t' = 0$. Applying the above results for time-refraction, and then transforming the frequency of the transmitted and reflected signals, ω_t and ω_r back to the lab frame S, we are then led to the following results [30, 47]

$$\omega_t^2 = \omega_i^2 \frac{(1 + s\beta n')^2}{(1 + s\beta)^2} + \omega_{p0}^2 \,, \quad \omega_r^2 = \omega_i^2 \frac{(1 - s\beta n')^2}{(1 - s\beta)^2} + \omega_{p0}^2 \tag{4.23}$$

where ω_i is the frequency of the incident signal, and $s = \pm 1$ pertains for parallel and anti-parallel propagation, such that $\mathbf{k_i} = s k_i \mathbf{e}_x$ and $k_i = \omega_i/c$. This shows that, in contrast with the previous time-refraction process occurring in the laboratory frame, the two frequencies of the transmitted and reflected signals can now be very different from each other. This is related to the fact that superluminal fronts break the symmetry of both space and time, and not just time. The physical consequences of this result are quite dramatic, because, for $\beta \sim 1$ one of these two frequencies becomes very large, according to the sign of s. This means that superluminal fronts can considerably enhance the time-refraction process, in terms of frequency and energy upshifts.

4.6 Time-Refraction Without Time

Let us now consider time-refraction in a completely different configuration. Superfluid light has been identified in recent years, associated with a photon beam (a laser beam) that propagates in a nonlinear Kerr medium [38–40]. The diffraction processes occurring in the transverse beam direction are very similar to those of a two-dimensional Bose-Einstein condensate, if we identify the axis of propagation as an effective temporal direction. This means that two transverse sections of the photon beam can be perceived as two different instants of time. When such a description is possible, the photon beam behaves as a two-dimensional superfluid. This opens new opportunities to study superfluid phenomena replacing the Bose-Einstein condensate experiments by much simpler nonlinear optical circuits.

An interesting version of time-refraction was recently identified in superfluid light, where temporal processes are absent, when the photon beam propagates in a static Kerr medium, but in the presence of a sharp discontinuity of the nonlinear susceptibility [42]. While photons propagate in a stationary beam along the medium, the existence of a sharp boundary between two regions of the medium, with two different nonlinear susceptibilities,

$\chi_1^{(3)}$ and $\chi_2^{(3)}$, leads to the possible excitation of diffraction waves in the perpendicular beam section. These diffraction waves satisfy a dispersion relation that is formally identical to that of Bogolioubov oscillations in a Bose-Einstein condensate, and can therefore be called *bogolons*. This is represented in Figure 4.6. The sharp spatial boundary between the two nonlinear regions can then be perceived as a temporal boundary, where the time variable is now the axial variable z divided by the velocity of light.

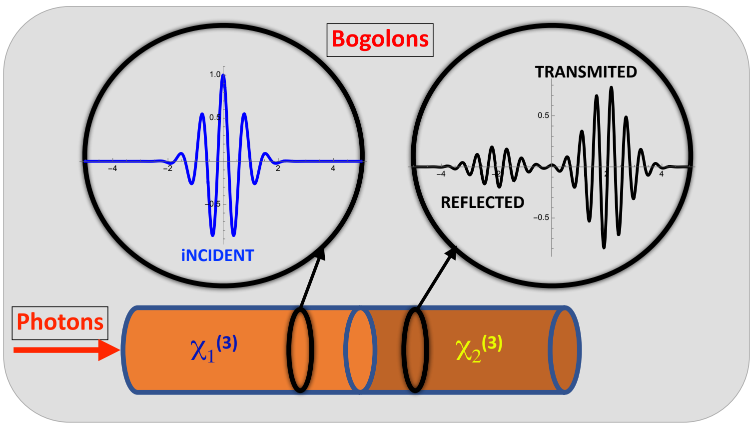

Figure 4.6: *Time-refraction in a static optical medium: bogolons instead of photons.*

In order to describe this process, we use the envelope equation for the electric field amplitude of the light beam, which is a nonlinear Schrödinger equation, but written in terms of BEC variables. In such a specific context, this nonlinear equation can more appropriately be called a Gross-Pitaevskii equation. For that purpose, we start from the electric field associated with a photon beam (or laser beam) propagation in the medium, as described by

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{E}(\mathbf{r}_\perp, z, t) \exp(ikz - \omega t) \, , \tag{4.24}$$

and use the paraxial approximation. After an appropriate change of variables [41], the paraxial equation describing the slow evolution of the beam amplitude can be written in the form

$$i\hbar \frac{\partial \mathcal{E}}{\partial t} = \left[-\frac{\hbar^2}{2m_k} \nabla_\perp^2 + V(\mathbf{r}_\perp, t) + g_k \left| \mathcal{E} \right|^2 \right] \mathcal{E} \, , \tag{4.25}$$

with $\nabla_\perp = (\partial^2/\partial x^2 + \partial^2/\partial y^2)$. This is formally identical to the Gross-Pitaevskii equation used to describe two-dimensional Bose-Einstein condensates in the mean-field approximation. But here, the role of the wavefunction is played by the electric field amplitude $\mathcal{E}$, and the time variable t is just a rescaled space variable, where z is divided by the speed of light, $t = z/c$. Furthermore, we have also introduced an effective photon mass m_k, and a coupling constant g_k, that describes the nonlinear optical properties of the medium, as defined by

$$m_k = \frac{\hbar k}{c} , \quad g_k = - \frac{\hbar \omega}{\sqrt{\epsilon(\omega)}} \chi^{(3)} , \tag{4.26}$$

where $\epsilon(\omega) = 1 + \chi^{(1)}(\omega)$ is the linear dielectric function. Here it is also assume that the axial wavenumber k is related to the wave frequency ω by the linear dispersion relation of the medium, such that $k^2 c^2 / \omega^2 = \epsilon(\omega)$. The external potential $V(\mathbf{r}_\perp, t)$ can be used to describe the boundary conditions of the beam and also eventual diffraction objects, but is ignored in the present discussion.

If we further explore the analogies with a two-dimensional condensate, we can make a *Madelung transformation*, such that $\mathcal{E} = \sqrt{\rho} \exp(i\theta)$, where the square of the beam amplitude (the beam energy) is now the fluid density ρ, and the gradient of the phase is the fluid velocity $\mathbf{v}$ in the perpendicular plane (x, y), as

$$\rho = |\mathcal{E}|^2 , \quad \mathbf{v} = \frac{\omega}{c} \frac{\nabla_\perp \theta}{\epsilon(\omega)} , \tag{4.27}$$

This allows us to derive from eq. (4.25) the equations for the photon fluid, which can be identified with *superfluid light*. In particular, if this fluid is perturbed by some optical irregularity, the resulting density perturbations will satisfy a dispersion relation that is formally identical to the Bogoliubov dispersion relation for a BEC, of the form

$$\Omega^2 = C_s^2 q^2 + \frac{\hbar^2 q^4}{4 m_k^2} , \quad C_s = \sqrt{\frac{g_k \rho_0}{m_k}} , \tag{4.28}$$

where C_s is the Bogoliubov speed, and ρ_0 is the unperturbed fluid density. This result is valid for density perturbations evolving with *frequency* Ω and wavevector $\mathbf{q}$, defined in the perpendicular plane (x, y). This is formally identical with what occurs in a two-dimensional Bose-Einstein condensate, but the quantity Ω is not a frequency in the usual sense, because the variable time t is not a physical time by represents the evolution along the z-axis. These perturbations are indeed diffraction waves, defined as beam intensity modulations in the perpendicular plane, and by analogy with the condensates can be called *bogolons*, because they satisfy identical dispersion relations.

Now, if at some point $z = z_0$ (which corresponds to an instant $t = t_0 \equiv z_0/c$ in temporal units) there is a sharp discontinuity of the nonlinear susceptibility, from an initial value $\chi_1^{(3)}$ to a different value $\chi_2^{(3)}$, in temporal units this can be seen as time-refraction. Therefore, a description similar to the above time-refraction can be used to relate the initial wave *frequency* Ω_1 to the final frequency Ω_2, because bogolons have to satisfy different dispersion relations for the same wavevector $\mathbf{q}$, where the Bogoliubov speed is different, because the value of g_k has changed. This is a process that is formally identical to time-refraction, but occurs in a static configuration where the abrupt changes take place along the paraxial direction z, and not in physical time. The elementary excitations of the medium are not photons, but bogolons, another kind of bosons. Instead of photons propagating in the axial direction they are now diffraction waves, defined in the perpendicular plane. Here, we can still define temporal reflection and transmission coefficients, similar but not identical to (4.4). In the present case of bogolons, we get [42]

$$R_b = \frac{(\alpha_b - 1)}{2\alpha_b} , \quad T_b = \frac{(\alpha_b + 1)}{2\alpha_b} , \tag{4.29}$$

which are very similar, but not identical to the temporal Fresnel's laws for photons, given by eqs. (4.3). The new parameter α_b is determined by the Bogoliubov dispersion, valid in the two sides of the medium (4.28), and is given by $\alpha_b = \Omega_2/\Omega_1$. Another difference with respect to photons is that the sum of the refraction and transmission coefficients is independent of the parameter α_b and equal to one, $R_b + T_b = 1$. Simulations of the Gross-Pitaevskii equation (4.25) confirm the validity of these results, and allow us to explore further the time-refraction mechanism in the nonlinear bogolon regime. We can also define a temporal beam-splitter for bogolons, as well as the possible formation of time-crystals, in this purely static problem. This illustrates the generally of the time-refraction concept, which can even survive the absence of temporal discontinuities.

4.7 Time-Refraction in QED

The time-refraction model can also be transposed to high energy QED, when the optical medium is replaced by quantum vacuum, which is a special kind of medium with no particles, simply made of virtual electron-positron pairs. This new configuration is relevant to the physics of ultra-intense lasers, in the multi-Petawatt regime [49, 50]. In this case, quantum vacuum is not the simple photon vacuum of quantum optics because it includes electrons and positrons, as described by the (quantized) Dirac's equation. This more complete view of QED vacuum contains photon states, as well as particle-pair states.

In order to discuss this last topic, we assume the case of pure vacuum, where intense electromagnetic fields are present and can be treated quasi-classically. Here, time-refraction is not related to a sudden change of refractive index, but to a sudden change of the vector potential, $\mathbf{A}(t)$, and leads to the excitation of particle-pairs. Such an elementary process can be described by a temporal Klein model.

Almost one century ago, Oscar Klein solved Dirac's equation for the case of a potential step $V(x)$, and showed that, for an incident electron, the reflection probability could be larger than one. This was called the *Klein paradox*, but is easily understood in terms of particle-pair formation, due to the singular electric field created by the potential step at $x = 0$. We can formulate a similar problem when the scalar potential is replaced by the vector potential and the step is defined, not in space, but in time. In this new problem, a singular electric field is created instantly at $t = 0$. Solution of Dirac's equation for this temporal problem is simple and leads, after some straightforward calculations, to the following expression for the electron-positron pair creation probability, for a given particle momentum $\mathbf{p}$ and spin polarization s [51]

$$P_{s,\mathbf{p}} \equiv \left| \langle 0,0 | \, \tilde{1}, \tilde{1} \rangle \right|^2_{s,\mathbf{p}} = \frac{a^2}{a^2 + \left(1 + \sqrt{1+a^2}\right)^2} \,, \tag{4.30}$$

where $a = eA/\sqrt{m^2 + p_\perp^2}$, is the normalized vector potential. Here the bra $\langle 0,0|$ represents the vacuum state with zero electrons and zero positrons, existing for $t < 0$, before the temporal transition. Similarly, the ket $\left|\tilde{1}, \tilde{1}\right\rangle$ represents the state with one electron with momentum $\mathbf{p}$ and spin s, and one positron with momentum $(-\mathbf{p})$ and spin $(-s)$, existing for $t > 0$, after the transition. This simple expression is valid for states such that $\mathbf{p} \simeq \mathbf{p}_\perp$, and momentum parallel to the vector potential is negligible. A more general expression can of course be derived for arbitrary values of the parallel momentum.

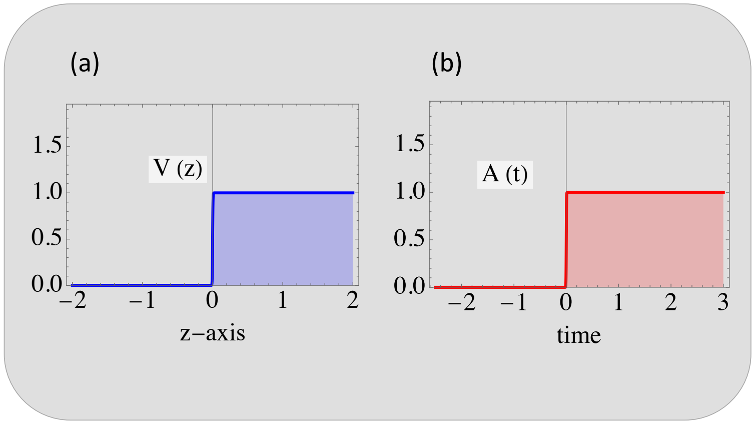

Figure 4.7: *Time-refraction in QED: (a) the usual scheme with a scalar potential step, as used in the formulation of the Klein paradox; (b) the temporal Klein model, where the scalar potential step is replaced by a vector potential time-step.*

This can be further generalized to arbitrary temporal variations of the vector potential $\mathbf{A}(t)$, described as a succession of infinitesimal time-steps, where the resulting electric field $\mathcal{E} = -\partial_t \mathbf{A}(t)$ can be that of an intense laser field [51]. This allows us to describe the pair production process in vacuum as a succession of elementary time-refraction events, which creates a kind of *temporal paradigm*, where temporal effects are dominant. This contrast with the usual *Schwinger paradigm*, which corresponds to a static electric field with infinite duration. This change of paradigm could be useful to study the case of ultra-short electric fields, such as those created by high-harmonic generation, was considered recently [52].

4.8 Conclusions

We have described the concept of time-refraction and shown that it is a first order universal process, at the same level as the usual (space) reflection studied in elementary optics and known for centuries. This means that time-refraction can be observed at the classical level, and concerns every light beam propagating in a non-stationary optical medium. This process is associated with time symmetry breaking, and envolves the excitation of a reflected signal. It means that, just as refraction occurring at the space boundary between two optical media, time-refraction occurs at a temporal boundary and produces two beams, the transmitted and the reflected beam. Expressions for time-transmission and time-reflection coefficients were given. They show that photon momentum is conserved but energy is not conserved, because the frequency changes at the temporal boundary in order to adjust to the new refractive index. We therefore conclude that time-refraction is characterized by the existence of a frequency shift.

Apart from these two signatures, time-reflection and frequency shifts, time-refraction also envolves a third property, when considered in the frame of quantum optics: the emission of photon-pairs from vacuum. In that sense, it can be related with the dynamical Casimir effect, a quantum process that is associated with the existence of an oscillating optical boundary. Although defined in different physical geometries, time-refraction and dynamical Casimir are in essence the same process at the elementary quantum level. But, time-refraction is more general, in the sense that it is independent of optical boundaries, and can occur in unbounded and uniform media. Under certain conditions, we can also establish a link with a third quantum vacuum process, the *Unruh effect*, by which a thermal spectrum is emitted by a particle with uniform acceleration in vacuum. Such a link is possible through the concept of *effective Unruh acceleration* proposed in [53], but is not completely clarified.

We have discussed the concept of a temporal device called temporal beam-splitter, which is made of a sequence of two inverse time-refraction events, and can be seen as the temporal counterpart of the optical beam-splitter, one of the most basic devices used in both classic and quantum optics experiments [41, 54]. More complicated temporal devices can also be imagined, such as time-cavities and time-crystals, through the addition of periodically spaced temporal beam-splitters. These new temporal arrays display resonant transmission and reflection properties, that can be described by simple *temporal Bragg laws.*

Although the basic concept of time-refraction is associated with sharp temporal boundaries, it can easily be extended to arbitrary non-stationary media, in analogy with the optics of inhomogeneous and stratified media. An arbitrary temporal evolution of the refraction index, $n(t)$ can be described as a succession of infinitesimal time-refraction events, and general expressions for time transmission and reflection coefficients can be derived, which resemble those of inhomogeneous optics. They give an alternative approach to the case of temporally periodic media, such as time-crystals. They can also be used to explore other time-symmetry breaking processes, such as expansion and quenching, and to examine the associated Kibble-Zurek mechanism, which have relevance in different areas of physics such as that of Bose-Einstein condensates [55–57], which were ignored here.

We have explored the novel results on superluminal ionization fronts, that can be created by short laser pulses using the flying focus concept. Different arrangements of this concept are being explored. These fronts can lead to significant frequency shifts, as well as to beam amplification, with relevance to future radiation sources. We have shown, using an appropriate reference frame called time-frame, that they can be described as a purely temporal event, identical to time-refraction. Furthermore, we have shown the resilience of the time-refraction concept by applying it to purely static processes involving diffraction waves in superfluid light. And finally, we have briefly discussed time-refraction in the context of laser QED, and its relation with the creation of electron-positron pairs by intense fields in vacuum. This variety of examples clearly demonstrates the significance, the wide range of applications, and the universality of time-refraction.

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4.9 References

[1] F. R. Morgenthaler, "Velocity Modulation of Electromag-netic Waves", *IRE Trans. Microwave Theory Tech.* **6**, 167 (1958).

[2] S.A. Akhmanov, A.P. Sukhorukov and A.S. Chirkin, "Nonstationary Phenomena and Space-Time Analogy in Nonlinear Optics", *Sov. Phys. JETP* **28**, 748 (1969).

[3] J.T. Mendonça and P.K. Shukla, "Time Refraction and Time Reflection: Two Basic Concepts", *Phys. Scripta* **65**, 160 (2002).

[4] B.W. Plansinis, W.R. Donaldson and G.P. Agrawal, "What is the Temporal Analog of Reflection and Refraction of Optical Beams?", *Phys. Rev. Lett.* **115**, 183901 (2015).

[5] J.T. Mendonça, A. Guerreiro and A.M. Martins, "Quantum Theory of Time Refraction", *Phys. Rev. A* **62**, 033805 (2000).

[6] V.V. Dodonov, A.B. Klimov, and D.E. Nikonov, "Quantum Phenomena in Nonstationary Media", *Phys. Rev. A* **47**, 4422 (1993).

[7] G. T. Moore, "Quantum Theory of the Electromagnetic Field in a Variable-length One-dimensional Cavity", J. Math. Phys. **11**, 2679 (1970).

[8] S.A. Fulling and P.C.W. Davies, "Radiation from a Moving Mirror in Two Dimensional Space-time: Conformal Anomaly", *Proc. R. Soc. Lond. A* **348**, 393 (1976).

[9] V. Dodonov, "Fifty Years of the Dynamical Casimir Effect", Physics **2**, 67 (2020).

[10] N. Westerberg, S. Cacciatori, F. Belgiorno, F. Dalla Piazza and D. Faccio, "Experimental Quantum Cosmology in Time-Dependent Optical Media", *New J. Phys.* **16**, 075003 (2014).

[11] C.M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J.R. Johansson, T. Duty, F. Nori, and P. Delsing, "Observation of the Dynamical Casimir Effect in a Superconducting Circuit", *Nature* **479**, 376 (2011).

[12] L. Yuan, and S. Fan, "Temporal Modulation Brings Metamaterials Into New Era", *Light Sci. Appl.* **11**, 173 (2022).

[13] C. Rizza, C. Castaldi and V. Galdi, "Sort-Pulsed Metamaterials", *Phys. Rev. Lett.* **128**, 257402 (2022).

[14] Y. Zhou, M.Z. Alam, M. Karimi, J. Upham, O. Reshef, C. Liu, A.E. Willner and R.W. Boyd, "Broadband Frequency Translation Through Time Refraction in an Epsilon-Near-Zero Material", *Nat. Commun.* **11**, 2180 (2020).

[15] O.Y. Long. K. Wang, A. Dutt and S. Fun, "Time Reflection and Refraction in Synthetic Frequency Dimension", *Phys. Rev. Research* **5**, L012046 (2023).

[16] M.R. Shcherbakov, K. Werner, Z. Fan, N. Talisa, E. Chowdhury and G. Shvets, "Photon Acceleration and Tunable Broadband Harmonics Generation in Nonlinear Time-dependent Metasurfaces", *Nat. Commun.* **10**, 1345 (2019); https://doi.org/10.1038/s41467-019-09313-8.

[17] J.E. Vázquez-Lozano and I. Liberal, "Shaping the Quantum Vacuum with Anisotropic Temporal Boundaries", *Nanophotonics* **12**, 539 (2023).

[18] E. Galiffi, R. Tirole, S. Yin, H. Li, S. Vezzoli, P.A. Huidobro, M.G. Silveirinha, R. Sapienza, A. Alù and J.B. Pendry, "Photonics of Time-Varying Media", *Adv. Photonics* **4**, 014002 (2022).

[19] H. Li, S. Yin and A. Alú, "Nonreciprocity and Faraday Rotation at Time Interfaces", *Phys. Rev. Lett.* **128**, 173901 (2022).

[20] E. Yablonovitch, "Self-phase Modulation and Short-Pulse Generation from Laser-Breakdown Plasmas", *Phys. Rev. A* **10**, 1888 (1974).

[21] S.P. Le Blanc, R. Sauerbrey, S.C. Rae and K. Burnett, “Spectral Blue Shifting of a Femtosecond Laser Pulse Propagating Through a High-Pressure Gas”, *J. Opt. Soc. Am. B* **10**, 1801 (1993).

[22] Y.M. Li and R. Fedosejevs, “Ionization-Induced Blue Shift of KrF Laser Pulses in an Underdense Plasma”, *Phys. Rev. E* **54**, 2166 (1996).

[23] H.J. Shin, D.G. Lee. Y.H. Cha, J.-H. Kim, K.H. Hong and C.H. Nam, “Nonadiabatic Blueshift of High-order Harmonics from Ar and Ne Atoms in an Intense Femtosecond Laser Field”, *Phys. REv. A* **63**, 053407 (2001).

[24] X. Xie, A.L. Cavalier and S.L. Johnson, “Self-Compression of Femtosecond Laser Pulses in Ambient Air Through Conical Radiation”, *Opt. Lett.* **48**, 5101 (2023).

[25] J.T. Mendonça, *Theory of Photon Acceleration*, IOP Publishing, Bristol (2001).

[26] E. Lustig, O. Segal, S. Saha, E. Bordo, S.N. Chowdhury, Y. Sharabi, A. Fleischer, A. Boltasseva, O. Cohen, V.M. Shalaev and M. Segev, “Time-Refraction Optics with Single Cycle Modulation”, *Nanophotonics* **12**, 2221 (2023).

[27] A. Giulietti and D. Giulietti, “Self-phase Modulation in Various Regimes of Intense Laser–Plasma Interactions”, *J. Plasma Phys.* **81**, 495810608 (2015).

[28] J.T. Mendonça, A.M. Martins and A. Guerreiro, “Quantum Theory of Time Refraction”, *Phys. Rev. A* **68**, 043801 (2003).

[29] V. Pacheco-Peña and N. Engheta, “Temporal Equivalent of the Brewster angle”, *Phys. Rev. B* **104**, 214308 (2021).

[30] J.T. Mendonça and A. Guerreiro, “Time refraction and the quantum properties of vacuum”, *Phys. Rev. A* **72**, 063805 (2005).

[31] V.L. Ginzburg, *Propagation of Electromagnetic Waves in Plasmas*, Gordon and Breach, New Your (1961).

[32] M. Kline and I.W. Kay, *Electromagnetic Theory and Geometric Optics*, John Wiley & Sons, New York (1965).

[33] F. Wilczek, “Quantum Time Crystals”, *Phys. Rev. Lett.* **109**, 160401 (2012).

[34] P. Bruno, “Impossibility of Spontaneously Rotating Time Crystals: A No-Go Theorem”, *Phys. Rev. Lett,* **111**, 070402 (2013).

[35] K. Sacha ans J. Zakrzewski, “Time Crystals: a review”, *Rep. Prog. Phys.* **81**, 016401 (2018).

[36] J.T. Mendonça and V.V. Dodonov, “Time Crystals in Ultracold Matter”, *J. Russian Laser Res.* **35**, 93 (2014).

[37] L.O. Silva and J.T. Mendonça, “Photon Acceleration in Superluminous and Accelerated Ionization Fronts”, *IEEE Trans. Plasma Sci.* **24**, 316 (1996).

[38] E.G. Khamis, A. Gammal, G.A. El, Yu.G. Gladuch and A.M. Kamchatnov, “Nonlinear Diffraction of Light Beams Propagating in Photorefractive Media with Embedded Reflecting Wire”, *Phys. Rev. A* **78**, 013829 (2008).

[39] C. Michel, O. Boughdad, M. Albert, P.-É. Larré, and M. Bellec, “Superfluid Motion and Drag-force Cancellation in a Fluid of Light”, *Nat. Commun.* **9**, 2108 (2018).

[40] Q. Fontaine et al., “Interferences Between Bogoliubov Excitations in Superfluids of Light”, *Phys. Rev. Research* **2**, 043297 (2020).

[41] J.T. Mendonça, *The Quantum Nature of Light*, IOP Publishing, Bristol (2022).

[42] J.T. Mendonça and I. Kourakis, “Propagation of Bogolons in Superfluid Light: Time-Like Effects in Non-Uniform Optical Media”, *Eur. Phys. J. Plus* **137**, 1269 (2022).

[43] Y. Pan, M.-I. Cohen and M. Segev, “Superluminal k-Gap Solitons in Nonlinear Photonic Time Crystals”, *Phys. Rev. Lett.* **130**, 233801 (2023).

[44] J.M. Dias, C. Stenz, N. Lopes, X. Badiche, F. Blasco, A. Dos Santos, L. O. Silva, A. Mysyrowicz, A. Antonetti and J.T. Mendonça, "Experimental Evidence of Photon Acceleration of Ultrashort Laser Pulses in Relativistic Ionization Fronts", *Phys. Rev. Lett.* **78**, 4773 (1997).
[45] A. Sainte-Marie, O. Gobert and F. Quéré, "Controlling the Velocity of Ultrashort Light Pulses in Vacuum through Spatio-temporal Couplings", *Optica* **4**, 1298 (2017).
[46] D.H. Froula, D. Turnbull, A.S. Davies et al., "Spatiotemporal Control of Laser Intensity", *Nature Photon* **12**, 262 (2018).
[47] A. Guerreiro, J.T. Mendonça and A.M. Martins, "New mechanism of vacuum radiation from non-accelerated moving boundaries", *J. Opt. B: Quantum Semiclass. Opt.* **7**, S69 (2005).
[48] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Elsevier Science, Amsterdam (2013).
[49] J.W.Yoon J W et al., "Realization of Laser Intensity over $10^{23} W/cm^2$", *Optica* **8**, 630 (2021).
[50] A. Fedotov, A. Ilderton, F. Karbstein, B. King, D. Seipt,H. Taya and G. Torgrimsson "Advances in QED with Intense Background Fields", *Phys. Rep.* **1010**, 1 (2023).
[51] J.T. Mendonça, "Temporal Klein Model for Particle-Pair Creation", *Symmetry* **13**, 1361 (2021); https://doi.org/10.3390/sym13081361
[52] J.T. Mendonça, "Particle-Pair Creation by High-Harmonic Laser Fields", *Phys. Scripta*, accepted (2023).
[53] J.T. Mendonça, G. Brodin and M. Marklund, "Vacuum Effects in a Vibrating Cavity: Time Refraction, Dynamical Casimir Effect, and Effective Unruh Acceleration", *Phys. Lett. A* **372**, 5621 (2008).
[54] C.C. Gerry and P.L. Knight, *Introductory Quantum Optics*, Cambridge University Press, Cambridge (2005).
[55] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, "Nonequilibrium Dynamics of Closed Interacting Quantum Systems", *Rev. Mod. Phys.***83**, 863 (2011)
[56] A. del Campo and W.H. Zurek, "Universality of Phase Transition Dynamics: Topological Defects from Symmetry Breaking" *Int. J. Mod. Phys. A* **29**, 1430018 (2014).
[57] J T Mendonça and A Gammal, "Time Symmetry Breaking in Bose–Einstein Condensates", *J. Phys. A: Math. Theor.* **50**, 355501 (2017).

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