CHAPTER 19 : Ben Goren, Kamal K. Barley, Sergei K. Suslov, Matrix Approach to Helicity States of Dirac Free Particles.

in : A. Dodonov and C.C.H. Ribeiro (Eds.), Proceedings of the Second International Workshop on Quantum Nonstationary Systems, pp .327 - 378 (LF Editorial, São Paulo, 2024). ISBN : 978-65-5563-446-4.

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[ORIGINAL ARTICLE: Ben Goren, Kamal Barley, Sergei K. Suslov, Matrix Approach to Helicity States of Dirac Free Particles. arXiv:2212.11994v2 [quant-ph]. https://doi.org/10.48550/arXiv.2212.11994.]

Abstract: We derive the free wave solutions of the Dirac equation from the viewpoint of matrix algebra. The concept of spin and the corresponding helicity states are analyzed in detail. This consideration may help the readers to study mathematical methods of relativistic quantum mechanics.

19.1 Abstract & 19.2 Introduction

We use elementary matrix algebra to derive the free wave solutions of the Dirac equation and examine the fundamental concepts of spin, polarization, and helicity states in details. This consideration can aid readers in studying the mathematical methods of relativistic quantum mechanics. We hope that our somewhat informal presentation may help beginners to enjoy the study of mathematics of relativistic quantum mechanics. To this end, we include important details of calculations which are usually omitted elsewhere. Computer algebra methods are useful for verification.

Wolfram Notebook:
This is our supplementary Mathematica notebook - always work in progress - do it yourself! (Please check for updates!)
We use below only basic Mathematica commands suitable for the beginners. You will be able to run each section and subsection independently from the others.

(Last modified on June 28, 2024; 1:55 PM, Arizona time.)

19.4. Matrix Algebra Plane Wave Solutions

Here we verify our identities (19.12) and (19.13)

Identity (19.12):

This is the first matrix in the l.h.s. of (19.12) in 4x4 form, say E=R>0:

```
{{m*c^2 - R, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2 - R, c*(p1 + I*p2), -c*p3},
  {c*p3, c*(p1 - I*p2), -m*c^2 - R, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2 - R}} // MatrixForm
```

$$\begin{pmatrix} c^2 m - R & 0 & c\,p_3 & c\,(p_1 - i\,p_2) \\ 0 & c^2 m - R & c\,(p_1 + i\,p_2) & -c\,p_3 \\ c\,p_3 & c\,(p_1 - i\,p_2) & -c^2 m - R & 0 \\ c\,(p_1 + i\,p_2) & -c\,p_3 & 0 & -c^2 m - R \end{pmatrix}$$

This is the second matrix in the l.h.s. of (19.12), when R→-R:

```
{{m*c^2 + R, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2 + R, c*(p1 + I*p2), -c*p3},
  {c*p3, c*(p1 - I*p2), -m*c^2 + R, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2 + R}} // MatrixForm
```

$$\begin{pmatrix} c^2 m + R & 0 & c\,p_3 & c\,(p_1 - i\,p_2) \\ 0 & c^2 m + R & c\,(p_1 + i\,p_2) & -c\,p_3 \\ c\,p_3 & c\,(p_1 - i\,p_2) & -c^2 m + R & 0 \\ c\,(p_1 + i\,p_2) & -c\,p_3 & 0 & -c^2 m + R \end{pmatrix}$$

Evaluating their product:

```
{{m*c^2 - R, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2 - R, c*(p1 + I*p2), -c*p3},
   {c*p3, c*(p1 - I*p2), -m*c^2 - R, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2 - R}}.
  {{m*c^2 + R, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2 + R, c*(p1 + I*p2), -c*p3},
   {c*p3, c*(p1 - I*p2), -m*c^2 + R, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2 + R}};
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} c^4 m^2 - R^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) & 0 & 0 & 0 \\ 0 & c^4 m^2 - R^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) & 0 & 0 \\ 0 & 0 & c^4 m^2 - R^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) & 0 \\ 0 & 0 & 0 & c^4 m^2 - R^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) \end{pmatrix}$$

This is the r.h.s. of (19.12) in 4x4 form.

Identity (19.13):

This is the matrices in the l.h.s. of (19.13):

```
{{m*c^2 + R, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2 + R, c*(p1 + I*p2), -c*p3},
  {c*p3, c*(p1 - I*p2), -m*c^2 - R, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2 - R}} // MatrixForm
```

$$\begin{pmatrix} c^2 m + R & 0 & c\,p_3 & c\,(p_1 - i\,p_2) \\ 0 & c^2 m + R & c\,(p_1 + i\,p_2) & -c\,p_3 \\ c\,p_3 & c\,(p_1 - i\,p_2) & -c^2 m - R & 0 \\ c\,(p_1 + i\,p_2) & -c\,p_3 & 0 & -c^2 m - R \end{pmatrix}$$

Evaluating the product in the l.h.s. of (19.13):

```
{{m*c^2 + R, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2 + R, c*(p1 + I*p2), -c*p3},
   {c*p3, c*(p1 - I*p2), -m*c^2 - R, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2 - R}}.
  {{m*c^2 + R, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2 + R, c*(p1 + I*p2), -c*p3},
   {c*p3, c*(p1 - I*p2), -m*c^2 - R, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2 - R}};
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \left(c^2 m + R\right)^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) & 0 & 0 & 0 \\ 0 & \left(c^2 m + R\right)^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) & 0 & 0 \\ 0 & 0 & \left(c^2 m + R\right)^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) & 0 \\ 0 & 0 & 0 & \left(c^2 m + R\right)^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right) \end{pmatrix}$$

This is the r.h.s. of (19.13).

19.5. Fermi's bi-spinors

Verification of Fermi's bi-spinors.

5.1 Equation (19.26):

This is the matrix in the l.h.s. of (19.26):

```
{{m*c^2, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2, c*(p1 + I*p2), -c*p3},
  {c*p3, c*(p1 - I*p2), -m*c^2, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2}} // MatrixForm
```

$$\begin{pmatrix} c^2\,m & 0 & c\,p_3 & c\,(p_1-i\,p_2) \\ 0 & c^2\,m & c\,(p_1+i\,p_2) & -c\,p_3 \\ c\,p_3 & c\,(p_1-i\,p_2) & -c^2\,m & 0 \\ c\,(p_1+i\,p_2) & -c\,p_3 & 0 & -c^2\,m \end{pmatrix}$$

This is Fermi's bi-spinor in the l.h.s. of (19.26); see also equation (27), on page 368, in Fermi's lectures:

```
{{(c*p3) / (R - m*c^2)}, {(c*(p1 + I*p2)) / (R - m*c^2)}, {1}, {0}} // MatrixForm
```

$$\begin{pmatrix} \frac{c\,p_3}{-c^2\,m+R} \\ \frac{c\,(p_1+i\,p_2)}{-c^2\,m+R} \\ 1 \\ 0 \end{pmatrix}$$

Evaluating their product:

```
{{m*c^2, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2, c*(p1 + I*p2), -c*p3},
   {c*p3, c*(p1 - I*p2), -m*c^2, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2}}.
  {{(c*p3) / (R - m*c^2)}, {(c*(p1 + I*p2)) / (R - m*c^2)}, {1}, {0}};
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{c\,R\,p_3}{c^2\,m-R} \\ -\frac{c\,R\,(p_1+i\,p_2)}{c^2\,m-R} \\ -\frac{c^2\left(c^2\,m^2-m\,R+p_1^2+p_2^2+p_3^2\right)}{c^2\,m-R} \\ 0 \end{pmatrix}$$

Substitution from (19.15):

```
% /. p1^2 + p2^2 + p3^2 → (R^2 - m^2*c^4) / c^2;
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{c\,R\,p_3}{c^2\,m-R} \\ -\frac{c\,R\,(p_1+i\,p_2)}{c^2\,m-R} \\ R \\ 0 \end{pmatrix}$$

This is the r.h.s. of (19.26).

5.2 In a similar fashion for the fourth Fermi's bi-spinor:

This is the matrix in the l.h.s. of (19.26), once again:

```
{{m*c^2, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2, c*(p1 + I*p2), -c*p3},
  {c*p3, c*(p1 - I*p2), -m*c^2, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2}} // MatrixForm
```

$$\begin{pmatrix} c^2\,m & 0 & c\,p_3 & c\,(p_1 - i\,p_2) \\ 0 & c^2\,m & c\,(p_1 + i\,p_2) & -c\,p_3 \\ c\,p_3 & c\,(p_1 - i\,p_2) & -c^2\,m & 0 \\ c\,(p_1 + i\,p_2) & -c\,p_3 & 0 & -c^2\,m \end{pmatrix}$$

This is the last bi-spinor (27), up to normalization, on page 368, in Fermi's lectures:

```
{{(c*(p1 - I*p2))/(R - m*c^2)}, {-(c*p3)/(R - m*c^2)}, {0}, {1}} // MatrixForm
```

$$\begin{pmatrix} \frac{c\,(p_1 - i\,p_2)}{-c^2\,m + R} \\ -\frac{c\,p_3}{-c^2\,m + R} \\ 0 \\ 1 \end{pmatrix}$$

Evaluating their product:

```
{{m*c^2, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2, c*(p1 + I*p2), -c*p3},
   {c*p3, c*(p1 - I*p2), -m*c^2, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2}}.
  {{(c*(p1 - I*p2))/(R - m*c^2)}, {-(c*p3)/(R - m*c^2)}, {0}, {1}};
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{c\,R\,(p_1 - i\,p_2)}{c^2\,m - R} \\ \frac{c\,R\,p_3}{c^2\,m - R} \\ 0 \\ -\frac{c^2\left(c^2\,m^2 - m\,R + p_1^2 + p_2^2 + p_3^2\right)}{c^2\,m - R} \end{pmatrix}$$

```
% /. p1^2 + p2^2 + p3^2 → (R^2 - m^2*c^4)/c^2;
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{c\,R\,(p_1 - i\,p_2)}{c^2\,m - R} \\ \frac{c\,R\,p_3}{c^2\,m - R} \\ 0 \\ R \end{pmatrix}$$

This is the last bi-spinor (27), on page 368, in Fermi's lectures multiplied by +R.

5.3 Fermi's bi-spinors are linearly dependent:

The (transposed) matrix of all four Fermi's bi-spinor (26-27) in Appendix B, page 368, in reverse order:

```
{{(c*(p1 - I*p2))/(R - m*c^2), -(c*p3)/(R - m*c^2), 0, 1},
  {(c*p3)/(R - m*c^2), (c*(p1 + I*p2))/(R - m*c^2), 1, 0},
  {0, 1, (c*(p1 - I*p2))/(R + m*c^2), -(c*p3)/(R + m*c^2)},
  {1, 0, (c*p3)/(R + m*c^2), (c*(p1 + I*p2))/(R + m*c^2)}} // MatrixForm
```

$$\begin{pmatrix} \frac{c\,(p_1-i\,p_2)}{-c^2\,m+R} & -\frac{c\,p_3}{-c^2\,m+R} & 0 & 1 \\ \frac{c\,p_3}{-c^2\,m+R} & \frac{c\,(p_1+i\,p_2)}{-c^2\,m+R} & 1 & 0 \\ 0 & 1 & \frac{c\,(p_1-i\,p_2)}{c^2\,m+R} & -\frac{c\,p_3}{c^2\,m+R} \\ 1 & 0 & \frac{c\,p_3}{c^2\,m+R} & \frac{c\,(p_1+i\,p_2)}{c^2\,m+R} \end{pmatrix}$$

```
Transpose[%] // MatrixForm
```

$$\begin{pmatrix} \frac{c\,(p_1-i\,p_2)}{-c^2\,m+R} & \frac{c\,p_3}{-c^2\,m+R} & 0 & 1 \\ -\frac{c\,p_3}{-c^2\,m+R} & \frac{c\,(p_1+i\,p_2)}{-c^2\,m+R} & 1 & 0 \\ 0 & 1 & \frac{c\,(p_1-i\,p_2)}{c^2\,m+R} & \frac{c\,p_3}{c^2\,m+R} \\ 1 & 0 & -\frac{c\,p_3}{c^2\,m+R} & \frac{c\,(p_1+i\,p_2)}{c^2\,m+R} \end{pmatrix}$$

```
Det[%];
FullSimplify[%]
```

$$\frac{\left(c^4\,m^2 - R^2 + c^2\left(p_1^2 + p_2^2 + p_3^2\right)\right)^2}{\left(-c^4\,m^2 + R^2\right)^2}$$

```
% /. p1^2 + p2^2 + p3^2 → (R^2 - m^2*c^4)/c^2
```

```
0
```

Thus the determinant of all four Fermi's bi-spinors (26-27), on page 368, is zero.

5.4 Verification of correct solutions in (19.18):

This is the matrix in the l.h.s. of (19.26), once again:

```
{{m*c^2, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2, c*(p1 + I*p2), -c*p3},
  {c*p3, c*(p1 - I*p2), -m*c^2, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2}} // MatrixForm
```

$$\begin{pmatrix} c^2\,m & 0 & c\,p_3 & c\,(p_1 - i\,p_2) \\ 0 & c^2\,m & c\,(p_1 + i\,p_2) & -c\,p_3 \\ c\,p_3 & c\,(p_1 - i\,p_2) & -c^2\,m & 0 \\ c\,(p_1 + i\,p_2) & -c\,p_3 & 0 & -c^2\,m \end{pmatrix}$$

These are two bi-spinors (19.19) for the positive energy eigenvalues:

```
{{1, 0}, {0, 1}, {(c*p3)/(R + m*c^2), (c*(p1 - I*p2))/(R + m*c^2)},
  {(c*(p1 + I*p2))/(R + m*c^2), -(c*p3)/(R + m*c^2)}} // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{c\,p_3}{c^2\,m+R} & \frac{c\,(p_1-i\,p_2)}{c^2\,m+R} \\ \frac{c\,(p_1+i\,p_2)}{c^2\,m+R} & -\frac{c\,p_3}{c^2\,m+R} \end{pmatrix}$$

Evaluating the product:

```
{{m*c^2, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2, c*(p1 + I*p2), -c*p3},
   {c*p3, c*(p1 - I*p2), -m*c^2, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2}}.
  {{1, 0}, {0, 1}, {(c*p3)/(R + m*c^2), (c*(p1 - I*p2))/(R + m*c^2)},
   {(c*(p1 + I*p2))/(R + m*c^2), -(c*p3)/(R + m*c^2)}};
FullSimplify[
  %];
```

```
% /. p1^2 + p2^2 + p3^2 → (R^2 - m^2*c^4)/c^2;
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} R & 0 \\ 0 & R \\ \frac{c\,R\,p_3}{c^2\,m+R} & \frac{c\,R\,(p_1-i\,p_2)}{c^2\,m+R} \\ \frac{c\,R\,(p_1+i\,p_2)}{c^2\,m+R} & -\frac{c\,R\,p_3}{c^2\,m+R} \end{pmatrix}$$

Those are our bi-spinors (19.19), up to normalization, or Fermi's first set (26), on page 368, multiplied by +R.

These are two bi-spinors (19.20) for the negative energy eigenvalues:

```
{{(c*p3)/(R + m*c^2), (c*(p1 - I*p2))/(R + m*c^2)},
  {(c*(p1 + I*p2))/(R + m*c^2), -(c*p3)/(R + m*c^2)}, {-1, 0}, {0, -1}} // MatrixForm
```

$$\begin{pmatrix} \frac{c\,p_3}{c^2\,m+R} & \frac{c\,(p_1-i\,p_2)}{c^2\,m+R} \\ \frac{c\,(p_1+i\,p_2)}{c^2\,m+R} & -\frac{c\,p_3}{c^2\,m+R} \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Evaluating the product:

```
FullSimplify[{{m*c^2, 0, c*p3, c*(p1 - I*p2)}, {0, m*c^2, c*(p1 + I*p2), -c*p3},
    {c*p3, c*(p1 - I*p2), -m*c^2, 0}, {c*(p1 + I*p2), -c*p3, 0, -m*c^2}}.
   {{(c*p3)/(R + m*c^2), (c*(p1 - I*p2))/(R + m*c^2)},
    {(c*(p1 + I*p2))/(R + m*c^2), -(c*p3)/(R + m*c^2)}, {-1, 0}, {0, -1}}];
```

```
% /. p1^2 + p2^2 + p3^2 → (R^2 - m^2*c^4)/c^2;
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{c\,R\,p_3}{c^2\,m+R} & -\frac{c\,R\,(p_1-i\,p_2)}{c^2\,m+R} \\ -\frac{c\,R\,(p_1+i\,p_2)}{c^2\,m+R} & \frac{c\,R\,p_3}{c^2\,m+R} \\ R & 0 \\ 0 & R \end{pmatrix}$$

Those are our bi-spinors (19.20) multiplied by -R.

19.6. Relativistic Helicity States

6.2 Nonrelativistic Helicity States

Equation (19.45): the first matrix.

```
{{Cos[θ/2]*Exp[-(I*ϕ)/2], Sin[θ/2]*Exp[-(I*ϕ)/2]},
 {Sin[θ/2]*Exp[(I*ϕ)/2], -Cos[θ/2]*Exp[(I*ϕ)/2]}} // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \end{pmatrix}$$

Equation (19.45): the second matrix.

```
{{Cos[θ/2]*Exp[(I*ϕ)/2], Sin[θ/2]*Exp[-(I*ϕ)/2]},
 {-Sin[θ/2]*Exp[(I*ϕ)/2], Cos[θ/2]*Exp[-(I*ϕ)/2]}} // MatrixForm
```

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \end{pmatrix}$$

Their product:

```
{{Cos[θ/2]*Exp[-(I*ϕ)/2], Sin[θ/2]*Exp[-(I*ϕ)/2]},
  {Sin[θ/2]*Exp[(I*ϕ)/2], -Cos[θ/2]*Exp[(I*ϕ)/2]}}.
 {{Cos[θ/2]*Exp[(I*ϕ)/2], Sin[θ/2]*Exp[-(I*ϕ)/2]},
  {-Sin[θ/2]*Exp[(I*ϕ)/2], Cos[θ/2]*Exp[-(I*ϕ)/2]}} // MatrixForm
```

$$\begin{pmatrix} \cos\left[\frac{\theta}{2}\right]^2 - \sin\left[\frac{\theta}{2}\right]^2 & 2\,e^{-i\phi}\cos\left[\frac{\theta}{2}\right]\sin\left[\frac{\theta}{2}\right] \\ 2\,e^{i\phi}\cos\left[\frac{\theta}{2}\right]\sin\left[\frac{\theta}{2}\right] & -\cos\left[\frac{\theta}{2}\right]^2 + \sin\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \cos[\theta] & e^{-i\phi}\sin[\theta] \\ e^{i\phi}\sin[\theta] & -\cos[\theta] \end{pmatrix}$$

This is the r.h.s of (19.45).

Checking (19.40) by matrix multiplication; both equations together as presented in the l.h.s. of (19.46):

```
{{Cos[θ], e^(-i ϕ) Sin[θ]},
 {e^(i ϕ) Sin[θ], -Cos[θ]}}.{{e^(-(i ϕ)/2) Cos[θ/2], -e^(-(i ϕ)/2) Sin[θ/2]},
 {e^((i ϕ)/2) Sin[θ/2], e^((i ϕ)/2) Cos[θ/2]}};
```

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is the r.h.s of (19.40&46).

Verification of equations (19.41): the spinor (19.38) and its Hermitian conjugate.

```
{{Cos[θ/2]*Exp[-(I*ϕ)/2]}, {Sin[θ/2]*Exp[(I*ϕ)/2]}} // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix}$$

```
Transpose[%];
% /. ϕ → -ϕ // MatrixForm
```

$$\begin{pmatrix} e^{\frac{i\phi}{2}} \text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}} \text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

For the first component of (19.41) with the first Pauli matrix:

```
{{0, 1}, {1, 0}} // MatrixForm
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} e^{\frac{i\phi}{2}} \text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}} \text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{i\phi}{2}} \text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}} \text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%]
```

```
{{Cos[ϕ] Sin[Θ]}}
```

[This is the first component of the polarization vector in (19 .36).]

In a similar fashion, for the second component of (19.41):

$$\begin{pmatrix} e^{\frac{i\phi}{2}} \text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}} \text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{i\phi}{2}} \text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}} \text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%]
```

```
{{Sin[Θ] Sin[ϕ]}}
```

For the third component of (19.41):

$$\begin{pmatrix} e^{\frac{i\phi}{2}} \text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}} \text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{i\phi}{2}} \text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}} \text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%]
```

```
{{Cos[Θ]}}
```

Thus, all three components of the first (19.41), with plus, are given by:

$$\left\{\left\{\begin{pmatrix} e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\end{pmatrix}\right\},\right.$$

$$\left\{\begin{pmatrix} e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\end{pmatrix}\cdot\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\cdot\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\end{pmatrix}\right\},$$

$$\left.\left\{\begin{pmatrix} e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\end{pmatrix}\right\}\right\};$$

```
FullSimplify[%];
MatrixForm[%]
```

$$\begin{pmatrix} (\,\text{Cos}[\phi]\,\text{Sin}[\theta]\,)\\ (\,\text{Sin}[\theta]\,\text{Sin}[\phi]\,)\\ (\,\text{Cos}[\theta]\,)\end{pmatrix}$$

In a similar fashion, for the spinor (19.39) and its Hermitian conjugate.

```
{{-Sin[θ/2] * Exp[-(I * ϕ)/2]}, {Cos[θ/2] * Exp[(I * ϕ)/2]}} // MatrixForm
```

$$\begin{pmatrix} -e^{-\frac{i\phi}{2}}\,\text{Sin}\left[\frac{\theta}{2}\right]\\ e^{\frac{i\phi}{2}}\,\text{Cos}\left[\frac{\theta}{2}\right]\end{pmatrix}$$

```
Transpose[%];
% /. ϕ → -ϕ // MatrixForm
```

$$\begin{pmatrix} -e^{\frac{i\phi}{2}}\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\,\text{Cos}\left[\frac{\theta}{2}\right]\end{pmatrix}$$

All three components of the second (19.41), with minus, are:

$$\left\{\left\{\begin{pmatrix} -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix} -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\\ e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\end{pmatrix}\right\},\right.$$

$$\left\{\begin{pmatrix} -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\end{pmatrix}\cdot\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\cdot\begin{pmatrix} -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\\ e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\end{pmatrix}\right\},$$

$$\left.\left\{\begin{pmatrix} -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix} -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]\\ e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]\end{pmatrix}\right\}\right\};$$

```
FullSimplify[%];
MatrixForm[%]
```

$$\begin{pmatrix} (-\text{Cos}[\phi]\,\text{Sin}[\theta]\,)\\ (-\text{Sin}[\theta]\,\text{Sin}[\phi]\,)\\ (-\text{Cos}[\theta]\,)\end{pmatrix}$$

Thus both equations (19.41) are verified.

6.6 Relativistic Helicity States

■ Helicity States

Checking action of the helicity operator (19.34) on all four bi-spinors in (19.104-105) and (19.131-132), $(\Sigma\cdot n)\mathcal{V}=?$:

```
{{Cos[θ], e^(-I ϕ) Sin[θ], 0, 0}, {e^(I ϕ) Sin[θ], -Cos[θ], 0, 0}, {0, 0, Cos[θ], e^(-I ϕ) Sin[θ]}, {0, 0, e^(I ϕ) Sin[θ], -Cos[θ]}} . {{e^(-I ϕ/2) Cos[θ/2], e^(-I ϕ/2) Sin[θ/2], e^(-I ϕ/2) η Sin[θ/2], e^(-I ϕ/2) η Cos[θ/2]}, {e^(I ϕ/2) Sin[θ/2], -e^(I ϕ/2) Cos[θ/2], -e^(I ϕ/2) η Cos[θ/2], e^(I ϕ/2) η Sin[θ/2]}, {e^(-I ϕ/2) η Cos[θ/2], -e^(-I ϕ/2) η Sin[θ/2], -e^(-I ϕ/2) Sin[θ/2], e^(-I ϕ/2) Cos[θ/2]}, {e^(I ϕ/2) η Sin[θ/2], e^(I ϕ/2) η Cos[θ/2], e^(I ϕ/2) Cos[θ/2], e^(I ϕ/2) Sin[θ/2]}};
```

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This implies that for the fist two bi-spinors, given by (19.104)-(19.105), the helicity signes are + and -. But for the last ones, given by (19.131)-(19.132), the helicity signes are - and +. Why? [Hint: Compare two solutions, say, in (19.159), and/or in (19.62) and (19.115), for E>0 and E<0, respectively.]

■ Hamiltonian and bi-spinors in matrix form

Hamiltonian $\mathbf{H}/\left(\mathbf{mc}^2\right)$ in (19.94) in parametrization (19.102):

```
{{1, 0, -(2 η Cos[θ])/(-1+η^2), -(2 e^(-I ϕ) η Sin[θ])/(-1+η^2)}, {0, 1, -(2 e^(I ϕ) η Sin[θ])/(-1+η^2), (2 η Cos[θ])/(-1+η^2)}, {-(2 η Cos[θ])/(-1+η^2), -(2 e^(-I ϕ) η Sin[θ])/(-1+η^2), -1, 0}, {-(2 e^(I ϕ) η Sin[θ])/(-1+η^2), (2 η Cos[θ])/(-1+η^2), 0, -1}} // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} \\ 0 & 1 & -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} \\ -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} & -1 & 0 \\ -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix}$$

Checking the first two bi-spinors in (19.104-105) by matrix multiplication:

```
{{1, 0, -(2 η Cos[θ])/(-1+η^2), -(2 e^(-I ϕ) η Sin[θ])/(-1+η^2)}, {0, 1, -(2 e^(I ϕ) η Sin[θ])/(-1+η^2), (2 η Cos[θ])/(-1+η^2)}, {-(2 η Cos[θ])/(-1+η^2), -(2 e^(-I ϕ) η Sin[θ])/(-1+η^2), -1, 0}, {-(2 e^(I ϕ) η Sin[θ])/(-1+η^2), (2 η Cos[θ])/(-1+η^2), 0, -1}} . {{e^(-I ϕ/2) Cos[θ/2], e^(-I ϕ/2) Sin[θ/2]}, {e^(I ϕ/2) Sin[θ/2], -e^(I ϕ/2) Cos[θ/2]}, {e^(-I ϕ/2) η Cos[θ/2], -e^(-I ϕ/2) η Sin[θ/2]}, {e^(I ϕ/2) η Sin[θ/2], e^(I ϕ/2) η Cos[θ/2]}};
```

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \end{pmatrix}$$

```
TensorProduct[{(1 - η^2) / (1 + η^2)}, {%}];
```

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix} \end{pmatrix}$$

Compare with (19.104-105) - that is correct for the positive energy eigenvalues E=R>0.

Checking the last two bi-spinors in (19.131-132) by matrix multiplication, with p → -p or $\eta \to -\eta$, in the Hamiltonian:

$$\begin{pmatrix} 1 & 0 & \frac{2\eta\,\text{Cos}[\theta]}{-1+\eta^2} & \frac{2e^{-i\phi}\eta\,\text{Sin}[\theta]}{-1+\eta^2} \\ 0 & 1 & \frac{2e^{i\phi}\eta\,\text{Sin}[\theta]}{-1+\eta^2} & -\frac{2\eta\,\text{Cos}[\theta]}{-1+\eta^2} \\ \frac{2\eta\,\text{Cos}[\theta]}{-1+\eta^2} & \frac{2e^{-i\phi}\eta\,\text{Sin}[\theta]}{-1+\eta^2} & -1 & 0 \\ \frac{2e^{i\phi}\eta\,\text{Sin}[\theta]}{-1+\eta^2} & -\frac{2\eta\,\text{Cos}[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ -e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \end{pmatrix}$$

```
TensorProduct[{(1 - η^2) / (1 + η^2)}, {%}];
```

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix} \end{pmatrix}$$

Compare with (19.131-132). The eigenvalue problem is verified for the negative energy eigenvalues $E = -R < 0$.

Let us combine both cases together, for the future consideration, and use only one Hamiltonian below. Then, checking all four bi-spinors in (19.104-105) and (19.131-132), with p → -p or $\eta \to -\eta$, by matrix multiplication, namely, $\mathbf{H}\mathcal{V} = \mathbf{R}\,\tilde{\mathcal{V}}$, one gets:

$$\begin{pmatrix} 1 & 0 & -\frac{2\eta\,\mathrm{Cos}[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\,\mathrm{Sin}[\theta]}{-1+\eta^2} \\ 0 & 1 & -\frac{2e^{i\phi}\eta\,\mathrm{Sin}[\theta]}{-1+\eta^2} & \frac{2\eta\,\mathrm{Cos}[\theta]}{-1+\eta^2} \\ -\frac{2\eta\,\mathrm{Cos}[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\,\mathrm{Sin}[\theta]}{-1+\eta^2} & -1 & 0 \\ -\frac{2e^{i\phi}\eta\,\mathrm{Sin}[\theta]}{-1+\eta^2} & \frac{2\eta\,\mathrm{Cos}[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Cos}\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]}{-1+\eta^2} \end{pmatrix}$$

This is our matrix $\mathbf{R}\,\tilde{\mathcal{V}} \,/\, \left(\mathbf{mc^2}\right)$; see (19.97).

```
Det[%];
```

```
FullSimplify[%]
```

$$\frac{\left(1+\eta^2\right)^6}{\left(-1+\eta^2\right)^4}$$

Recall the matrix $\mathcal{V}$ once again:

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\mathrm{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\mathrm{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\mathrm{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
Det[%];
```

```
FullSimplify[%]
```

$$\left(1+\eta^2\right)^2$$

Thus, all four bi-spinors in (19.104-105), the first two columns, and in (19.131-132), with p → -p or $\eta \to -\eta$ in the last two ones, are linearly independent.

Practice: multiplication of matrix $\mathbf{R} \,/\, \left(\mathbf{mc^2}\right) * \tilde{\mathcal{V}}$ by the reciprocal of the energy constant:

```
TensorProduct[{(1 - η^2) / (1 + η^2)},
```

$$\left\{\begin{pmatrix} -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} \end{pmatrix}\right\};$$

```
(* Matrix ν̃: *) FullSimplify[%] // MatrixForm
```

$$\left\{\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix}\right\}$$

Here, the first two columns correspond to positive energy eigenvalues and the last two - to the negative energy eigenvalues.

Checking all four bi-spinors in (19.104-105) and (19.131-132), with p → -p or $\eta \to -\eta$, by matrix multiplication, in the second relation (19.97), **H $\tilde{\nu}$ = Rν :**

$$\begin{pmatrix} 1 & 0 & -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} \\ 0 & 1 & -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} \\ -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} & -1 & 0 \\ -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\left(\eta+\eta^3\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\left(1+\eta^2\right)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} \end{pmatrix}$$

This is the matrix **Rν / (mc²)**; see (19.97).

Practice: multiplication of matrix **R / (mc²) * ν** by the reciprocal of the energy constant:

```
TensorProduct[{(1 - η^2) / (1 + η^2)},
```

$$\left\{\begin{pmatrix} -\frac{e^{-\frac{i\phi}{2}}(1+\eta^2)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}(1+\eta^2)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta(1+\eta^2)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta(1+\eta^2)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}(1+\eta^2)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}(1+\eta^2)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}(\eta+\eta^3)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}(\eta+\eta^3)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{-\frac{i\phi}{2}}\eta(1+\eta^2)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta(1+\eta^2)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}(1+\eta^2)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}(1+\eta^2)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}(\eta+\eta^3)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}(\eta+\eta^3)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}(1+\eta^2)\cos\left[\frac{\theta}{2}\right]}{-1+\eta^2} & \frac{e^{\frac{i\phi}{2}}(1+\eta^2)\sin\left[\frac{\theta}{2}\right]}{-1+\eta^2} \end{pmatrix}\right\}];$$

```
(* Matrix 𝒱: *) FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix}$$

Squared Hamiltonian is diagonal (19.95), $\left(H/\left(mc^2\right)\right)^2 = R^2\,\mathcal{I}$:

$$\begin{pmatrix} 1 & 0 & -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} \\ 0 & 1 & -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} \\ -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} & -1 & 0 \\ -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} \\ 0 & 1 & -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} \\ -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} & -1 & 0 \\ -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \frac{(1+\eta^2)^2}{(-1+\eta^2)^2} & 0 & 0 & 0 \\ 0 & \frac{(1+\eta^2)^2}{(-1+\eta^2)^2} & 0 & 0 \\ 0 & 0 & \frac{(1+\eta^2)^2}{(-1+\eta^2)^2} & 0 \\ 0 & 0 & 0 & \frac{(1+\eta^2)^2}{(-1+\eta^2)^2} \end{pmatrix}$$

■ Inverses

We also need $\mathcal{V}^{-1}$:

```
Inverse[
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix}];$$

```
(* 𝒱^-1: *) FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \end{pmatrix}$$

Checking $\mathcal{V}\,\mathcal{V}^{-1} = \mathcal{I}$:

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And finding $\tilde{\mathcal{V}}^{-1}$:

```
Inverse[ ( e^-(i ϕ/2) Cos[θ/2]      e^-(i ϕ/2) Sin[θ/2]      -e^-(i ϕ/2) η Sin[θ/2]   -e^-(i ϕ/2) η Cos[θ/2]
           e^(i ϕ/2) Sin[θ/2]       -e^(i ϕ/2) Cos[θ/2]      e^(i ϕ/2) η Cos[θ/2]     -e^(i ϕ/2) η Sin[θ/2]
           e^-(i ϕ/2) η Cos[θ/2]    -e^-(i ϕ/2) η Sin[θ/2]   -e^-(i ϕ/2) Sin[θ/2]     e^-(i ϕ/2) Cos[θ/2]
           e^(i ϕ/2) η Sin[θ/2]     e^(i ϕ/2) η Cos[θ/2]     e^(i ϕ/2) Cos[θ/2]       e^(i ϕ/2) Sin[θ/2] ) ];
```

```
(* 𝒱~^-1: *) FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \end{pmatrix}$$

Checking $\tilde{\mathcal{V}} \cdot \tilde{\mathcal{V}}^{-1} = \mathcal{I}$:

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

■ Decompositions of the Hamiltonian

Decompositions (19.99): $\mathbf{R}\,\tilde{\mathcal{V}}\cdot\mathcal{V}^{-1} = \mathbf{R}\mathcal{V}\cdot\tilde{\mathcal{V}}^{-1} = \mathbf{H}$

$\tilde{\mathcal{V}}\cdot\mathcal{V}^{-1}$:

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -1+\frac{2}{1+\eta^2} & 0 & \frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} & \frac{2e^{-i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} \\ 0 & -1+\frac{2}{1+\eta^2} & \frac{2e^{i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} & -\frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} \\ \frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} & \frac{2e^{-i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} & 1-\frac{2}{1+\eta^2} & 0 \\ \frac{2e^{i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} & -\frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} & 0 & 1-\frac{2}{1+\eta^2} \end{pmatrix}$$

$\mathcal{V}\cdot\tilde{\mathcal{V}}^{-1}$:

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} \\ -\frac{e^{\frac{i\phi}{2}}\eta\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & -\frac{e^{-\frac{i\phi}{2}}\eta\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{\frac{i\phi}{2}}\cos\left[\frac{\theta}{2}\right]}{1+\eta^2} & \frac{e^{-\frac{i\phi}{2}}\sin\left[\frac{\theta}{2}\right]}{1+\eta^2} \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -1+\frac{2}{1+\eta^2} & 0 & \frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} & \frac{2e^{-i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} \\ 0 & -1+\frac{2}{1+\eta^2} & \frac{2e^{i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} & -\frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} \\ \frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} & \frac{2e^{-i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} & 1-\frac{2}{1+\eta^2} & 0 \\ \frac{2e^{i\phi}\eta\,\mathrm{Sin}[\Theta]}{1+\eta^2} & -\frac{2\eta\,\mathrm{Cos}[\Theta]}{1+\eta^2} & 0 & 1-\frac{2}{1+\eta^2} \end{pmatrix}$$

Here, both matrix products are the same:

```
Simplify[% - %%] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Simplify[TensorProduct[{(1 + η^2) / (1 - η^2)}, {{-1 + 2/(1+η^2), 0, 2 η Cos[θ]/(1+η^2), 2 E^(-I ϕ) η Sin[θ]/(1+η^2)}, {0, -1 + 2/(1+η^2), 2 E^(I ϕ) η Sin[θ]/(1+η^2), -2 η Cos[θ]/(1+η^2)}, {2 η Cos[θ]/(1+η^2), 2 E^(-I ϕ) η Sin[θ]/(1+η^2), 1 - 2/(1+η^2), 0}, {2 E^(I ϕ) η Sin[θ]/(1+η^2), -2 η Cos[θ]/(1+η^2), 0, 1 - 2/(1+η^2)}}]] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} \\ 0 & 1 & -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} \\ -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} & -1 & 0 \\ -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix} \end{pmatrix}$$

This is Dirac's Hamiltonian in the units (19.102).

Indeed, recall that the Hamiltonian $\mathbf{H} / (\mathbf{mc^2})$ in (19.94) in parametrization (19.102) has the identical form:

```
{{1, 0, -2 η Cos[θ]/(-1+η^2), -2 E^(-I ϕ) η Sin[θ]/(-1+η^2)}, {0, 1, -2 E^(I ϕ) η Sin[θ]/(-1+η^2), 2 η Cos[θ]/(-1+η^2)}, {-2 η Cos[θ]/(-1+η^2), -2 E^(-I ϕ) η Sin[θ]/(-1+η^2), -1, 0}, {-2 E^(I ϕ) η Sin[θ]/(-1+η^2), 2 η Cos[θ]/(-1+η^2), 0, -1}} // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} \\ 0 & 1 & -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} \\ -\frac{2\eta\cos[\theta]}{-1+\eta^2} & -\frac{2e^{-i\phi}\eta\sin[\theta]}{-1+\eta^2} & -1 & 0 \\ -\frac{2e^{i\phi}\eta\sin[\theta]}{-1+\eta^2} & \frac{2\eta\cos[\theta]}{-1+\eta^2} & 0 & -1 \end{pmatrix}$$

```
Simplify[% - {{1, 0, -2 η Cos[θ]/(-1+η^2), -2 E^(-I ϕ) η Sin[θ]/(-1+η^2)}, {0, 1, -2 E^(I ϕ) η Sin[θ]/(-1+η^2), 2 η Cos[θ]/(-1+η^2)}, {-2 η Cos[θ]/(-1+η^2), -2 E^(-I ϕ) η Sin[θ]/(-1+η^2), -1, 0}, {-2 E^(I ϕ) η Sin[θ]/(-1+η^2), 2 η Cos[θ]/(-1+η^2), 0, -1}}] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Convenient form of $\tilde{\gamma} \cdot \gamma^{-1}$:

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}.$$

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 1-\eta^2 & 0 & 2\eta\,\text{Cos}[\theta] & 2e^{-i\phi}\eta\,\text{Sin}[\theta] \\ 0 & 1-\eta^2 & 2e^{i\phi}\eta\,\text{Sin}[\theta] & -2\eta\,\text{Cos}[\theta] \\ 2\eta\,\text{Cos}[\theta] & 2e^{-i\phi}\eta\,\text{Sin}[\theta] & -1+\eta^2 & 0 \\ 2e^{i\phi}\eta\,\text{Sin}[\theta] & -2\eta\,\text{Cos}[\theta] & 0 & -1+\eta^2 \end{pmatrix}$$

Convenient form of $\mathcal{V}\cdot\tilde{\mathcal{V}}^{-1}$:

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}.$$

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 1-\eta^2 & 0 & 2\eta\,\text{Cos}[\theta] & 2e^{-i\phi}\eta\,\text{Sin}[\theta] \\ 0 & 1-\eta^2 & 2e^{i\phi}\eta\,\text{Sin}[\theta] & -2\eta\,\text{Cos}[\theta] \\ 2\eta\,\text{Cos}[\theta] & 2e^{-i\phi}\eta\,\text{Sin}[\theta] & -1+\eta^2 & 0 \\ 2e^{i\phi}\eta\,\text{Sin}[\theta] & -2\eta\,\text{Cos}[\theta] & 0 & -1+\eta^2 \end{pmatrix}$$

```
Simplify[% - %%] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

19.7. Polarization Density Matrices

Here we verify the projection operators, describing the polarization of spin one-half particles, introduced by Mitchel and Wightman (Figure 19.3, Ref. [36]), by direct matrix multiplications.

Case E>0 and λ=1/2. Identities (19.126)-(19.130):

Let us construct the tensor product in the l.h.s. of (19.124). This is our bi-spinor (19.104):

```
{Cos[θ/2]*Exp[-(I*ϕ)/2], Sin[θ/2]*Exp[(I*ϕ)/2],
  η*Cos[θ/2]*Exp[-(I*ϕ)/2], η*Sin[θ/2]*Exp[(I*ϕ)/2]} // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\,\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\,\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is its complex conjugate:

```
% /. ϕ → -ϕ // MatrixForm
```

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\,\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\,\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is the Dirac conjugate:

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}.{Cos[θ/2]*Exp[(I*ϕ)/2],
  Sin[θ/2]*Exp[-(I*ϕ)/2], η*Cos[θ/2]*Exp[(I*ϕ)/2], η*Sin[θ/2]*Exp[-(I*ϕ)/2]}
```

$$\left\{e^{\frac{i\phi}{2}}\,\text{Cos}\left[\frac{\theta}{2}\right],\ e^{-\frac{i\phi}{2}}\,\text{Sin}\left[\frac{\theta}{2}\right],\ -e^{\frac{i\phi}{2}}\,\eta\,\text{Cos}\left[\frac{\theta}{2}\right],\ -e^{-\frac{i\phi}{2}}\,\eta\,\text{Sin}\left[\frac{\theta}{2}\right]\right\}$$

```
% // MatrixForm
```

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\,\text{Sin}\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\,\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ -e^{-\frac{i\phi}{2}}\,\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is their tensor product (19.130):

```
TensorProduct[{Cos[θ/2]*Exp[-(I*ϕ)/2], Sin[θ/2]*Exp[(I*ϕ)/2],
   η*Cos[θ/2]*Exp[-(I*ϕ)/2], η*Sin[θ/2]*Exp[(I*ϕ)/2]},
  {e^(iϕ/2) Cos[θ/2], e^(-iϕ/2) Sin[θ/2], -η e^(iϕ/2) Cos[θ/2], -η e^(-iϕ/2) Sin[θ/2]}] // MatrixForm
```

$$\begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & -\eta\cos^2\frac{\theta}{2} & -e^{-i\phi}\eta\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} & -e^{i\phi}\eta\cos\frac{\theta}{2}\sin\frac{\theta}{2} & -\eta\sin^2\frac{\theta}{2} \\ \eta\cos^2\frac{\theta}{2} & e^{-i\phi}\eta\cos\frac{\theta}{2}\sin\frac{\theta}{2} & -\eta^2\cos^2\frac{\theta}{2} & -e^{-i\phi}\eta^2\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\eta\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \eta\sin^2\frac{\theta}{2} & -e^{i\phi}\eta^2\cos\frac{\theta}{2}\sin\frac{\theta}{2} & -\eta^2\sin^2\frac{\theta}{2} \end{pmatrix}$$

```
% /. Cos[θ/2] Sin[θ/2] → 1/2 * Sin[θ] // MatrixForm
```

$$\begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\phi}\sin[\theta] & -\eta\cos^2\frac{\theta}{2} & -\frac{1}{2}e^{-i\phi}\eta\sin[\theta] \\ \frac{1}{2}e^{i\phi}\sin[\theta] & \sin^2\frac{\theta}{2} & -\frac{1}{2}e^{i\phi}\eta\sin[\theta] & -\eta\sin^2\frac{\theta}{2} \\ \eta\cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\phi}\eta\sin[\theta] & -\eta^2\cos^2\frac{\theta}{2} & -\frac{1}{2}e^{-i\phi}\eta^2\sin[\theta] \\ \frac{1}{2}e^{i\phi}\eta\sin[\theta] & \eta\sin^2\frac{\theta}{2} & -\frac{1}{2}e^{i\phi}\eta^2\sin[\theta] & -\eta^2\sin^2\frac{\theta}{2} \end{pmatrix}$$

This is the matrix in (19.130).

This is the matrix in (19.126):

```
{{1, 0, -η*Cos[θ], -η*Sin[θ]*Exp[-I*ϕ]},
  {0, 1, -η*Sin[θ]*Exp[I*ϕ], η*Cos[θ]}, {η*Cos[θ], η*Sin[θ]*Exp[-I*ϕ], -η^2, 0},
  {η*Sin[θ]*Exp[I*ϕ], -η*Cos[θ], 0, -η^2}} // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & -\eta\cos[\theta] & -e^{-i\phi}\eta\sin[\theta] \\ 0 & 1 & -e^{i\phi}\eta\sin[\theta] & \eta\cos[\theta] \\ \eta\cos[\theta] & e^{-i\phi}\eta\sin[\theta] & -\eta^2 & 0 \\ e^{i\phi}\eta\sin[\theta] & -\eta\cos[\theta] & 0 & -\eta^2 \end{pmatrix}$$

This is the matrix in our equation just before (19.130):

```
{{(Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, (1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ], -η, 0},
  {(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, 0, -η},
  {η, 0, (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, -(1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ]},
  {0, η, -(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2}} // MatrixForm
```

$$\begin{pmatrix} \cos^2\frac{\theta}{2} - \eta^2\sin^2\frac{\theta}{2} & \frac{1}{2}e^{-i\phi}\left(1+\eta^2\right)\sin[\theta] & -\eta & 0 \\ \frac{1}{2}e^{i\phi}\left(1+\eta^2\right)\sin[\theta] & -\eta^2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} & 0 & -\eta \\ \eta & 0 & -\eta^2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} & -\frac{1}{2}e^{-i\phi}\left(1+\eta^2\right)\sin[\theta] \\ 0 & \eta & -\frac{1}{2}e^{i\phi}\left(1+\eta^2\right)\sin[\theta] & \cos^2\frac{\theta}{2} - \eta^2\sin^2\frac{\theta}{2} \end{pmatrix}$$

This is multiplication of the last two matrices

```
{{1, 0, -η*Cos[θ], -η*Sin[θ]*Exp[-I*ϕ]}, {0, 1, -η*Sin[θ]*Exp[I*ϕ], η*Cos[θ]},
   {η*Cos[θ], η*Sin[θ]*Exp[-I*ϕ], -η^2, 0}, {η*Sin[θ]*Exp[I*ϕ], -η*Cos[θ], 0, -η^2}} .
  {{(Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, (1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ], -η, 0},
   {(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, 0, -η},
   {η, 0, (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, -(1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ]},
   {0, η, -(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2}};
```

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{1}{2}\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \eta\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ -\frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \eta^2\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

Now we compare this result with (19.130):

$$\begin{pmatrix} -\frac{1}{2}\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \eta\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ -\frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \eta^2\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix} -$$

$$\left(1-\eta^2\right)*\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\text{Sin}[\theta] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\text{Sin}[\theta] & \text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & -\eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the first matrix identity (19.124) is verified.

Case E>0 and λ=-1/2. The second identity (19.124):

This is the second tensor product in (19.124), for the bi-spinor (19.105):

```
TensorProduct[{Sin[θ/2]*Exp[-(I*ϕ)/2], -Cos[θ/2]*Exp[(I*ϕ)/2],
   -η*Sin[θ/2]*Exp[-(I*ϕ)/2], η*Cos[θ/2]*Exp[(I*ϕ)/2]},
  {e^(iϕ/2) Sin[θ/2], -e^(-iϕ/2) Cos[θ/2], η e^(iϕ/2) Sin[θ/2], -η e^(-iϕ/2) Cos[θ/2]}] // MatrixForm
```

$$\begin{pmatrix} \text{Sin}\left[\frac{\theta}{2}\right]^2 & -e^{-i\phi}\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -e^{-i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] \\ -e^{i\phi}\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \text{Cos}\left[\frac{\theta}{2}\right]^2 & -e^{i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \\ -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & e^{-i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & e^{-i\phi}\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & e^{i\phi}\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

```
% /. Cos[θ/2] Sin[θ/2] → 1/2 * Sin[θ] // MatrixForm
```

$$\begin{pmatrix} \text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\,\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\,\text{Sin}[\theta] & \text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \\ -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

This is the second polarization density matrix in the l.h.s. of (19.124).

The matrix in (19.126):

```
{{1, 0, -η*Cos[θ], -η*Sin[θ]*Exp[-I*ϕ]},
  {0, 1, -η*Sin[θ]*Exp[I*ϕ], η*Cos[θ]}, {η*Cos[θ], η*Sin[θ]*Exp[-I*ϕ], -η^2, 0},
  {η*Sin[θ]*Exp[I*ϕ], -η*Cos[θ], 0, -η^2}} // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & -\eta\,\text{Cos}[\theta] & -e^{-i\phi}\,\eta\,\text{Sin}[\theta] \\ 0 & 1 & -e^{i\phi}\,\eta\,\text{Sin}[\theta] & \eta\,\text{Cos}[\theta] \\ \eta\,\text{Cos}[\theta] & e^{-i\phi}\,\eta\,\text{Sin}[\theta] & -\eta^2 & 0 \\ e^{i\phi}\,\eta\,\text{Sin}[\theta] & -\eta\,\text{Cos}[\theta] & 0 & -\eta^2 \end{pmatrix}$$

should be multiplied by the matrix in the next equation to (19.134):

```
{{(Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, -(1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ], η, 0},
  {-(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, 0, η},
  {-η, 0, (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, (1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ]},
  {0, -η, (1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2}} // MatrixForm
```

$$\begin{pmatrix} -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 + \text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}\,e^{-i\phi}\left(1+\eta^2\right)\text{Sin}[\theta] & \eta & 0 \\ -\frac{1}{2}\,e^{i\phi}\left(1+\eta^2\right)\text{Sin}[\theta] & \text{Cos}\left[\frac{\theta}{2}\right]^2 - \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & 0 & \eta \\ -\eta & 0 & \text{Cos}\left[\frac{\theta}{2}\right]^2 - \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}\,e^{-i\phi}\left(1+\eta^2\right)\text{Sin}[\theta] \\ 0 & -\eta & \frac{1}{2}\,e^{i\phi}\left(1+\eta^2\right)\text{Sin}[\theta] & -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 + \text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

Indeed

```
{{1, 0, -η*Cos[θ], -η*Sin[θ]*Exp[-I*ϕ]}, {0, 1, -η*Sin[θ]*Exp[I*ϕ], η*Cos[θ]},
   {η*Cos[θ], η*Sin[θ]*Exp[-I*ϕ], -η^2, 0}, {η*Sin[θ]*Exp[I*ϕ], -η*Cos[θ], 0, -η^2}} .
  {{(Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, -(1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ], η, 0},
   {-(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, 0, η},
   {-η, 0, (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, (1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ]},
   {0, -η, (1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2}};
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\left(-1+\eta^2\right)(-1+\text{Cos}[\theta]) & \frac{1}{2}\,e^{-i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\,\eta\left(-1+\eta^2\right)(-1+\text{Cos}[\theta]) & \frac{1}{2}\,e^{-i\phi}\,\eta\left(-1+\eta^2\right)\text{Sin}[\theta] \\ \frac{1}{2}\,e^{i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & -\frac{1}{2}\left(-1+\eta^2\right)(1+\text{Cos}[\theta]) & \frac{1}{2}\,e^{i\phi}\,\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & -\frac{1}{2}\,\eta\left(-1+\eta^2\right)(1+\text{Cos}[\theta]) \\ \eta\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}\,e^{-i\phi}\,\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \eta^2\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}\,e^{-i\phi}\,\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}\,e^{i\phi}\,\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\,\eta\left(-1+\eta^2\right)(1+\text{Cos}[\theta]) & -\frac{1}{2}\,e^{i\phi}\,\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\,\eta^2\left(-1+\eta^2\right)(1+\text{Cos}[\theta]) \end{pmatrix}$$

Checking the matrix identity:

$$\begin{pmatrix} \frac{1}{2}(-1+\eta^2)(-1+\text{Cos}[\theta]) & \frac{1}{2}e^{-i\phi}(-1+\eta^2)\text{Sin}[\theta] & \frac{1}{2}\eta(-1+\eta^2)(-1+\text{Cos}[\theta]) & \frac{1}{2}e^{-i\phi}\eta(-1+\eta^2)\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}(-1+\eta^2)\text{Sin}[\theta] & -\frac{1}{2}(-1+\eta^2)(1+\text{Cos}[\theta]) & \frac{1}{2}e^{i\phi}\eta(-1+\eta^2)\text{Sin}[\theta] & -\frac{1}{2}\eta(-1+\eta^2)(1+\text{Cos}[\theta]) \\ \eta(-1+\eta^2)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta(-1+\eta^2)\text{Sin}[\theta] & \eta^2(-1+\eta^2)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta^2(-1+\eta^2)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta(-1+\eta^2)\text{Sin}[\theta] & \frac{1}{2}\eta(-1+\eta^2)(1+\text{Cos}[\theta]) & -\frac{1}{2}e^{i\phi}\eta^2(-1+\eta^2)\text{Sin}[\theta] & \frac{1}{2}\eta^2(-1+\eta^2)(1+\text{Cos}[\theta]) \end{pmatrix} +$$

$$(-1+\eta^2) * \begin{pmatrix} \text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\text{Sin}[\theta] & \text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \\ -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the second matrix identity (19.124) is verified by Mathematica.

Case E<0 and λ=-1/2. Identities (19.125), (19.134-135):

This is the matrix in (19.134):

```
{{η^2, 0, -η*Cos[θ], -η*Sin[θ]*Exp[-I*ϕ]},
  {0, η^2, -η*Sin[θ]*Exp[I*ϕ], η*Cos[θ]}, {η*Cos[θ], η*Sin[θ]*Exp[-I*ϕ], -1, 0},
  {η*Sin[θ]*Exp[I*ϕ], -η*Cos[θ], 0, -1}} // MatrixForm
```

$$\begin{pmatrix} \eta^2 & 0 & -\eta\,\text{Cos}[\theta] & -e^{-i\phi}\eta\,\text{Sin}[\theta] \\ 0 & \eta^2 & -e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Cos}[\theta] \\ \eta\,\text{Cos}[\theta] & e^{-i\phi}\eta\,\text{Sin}[\theta] & -1 & 0 \\ e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Cos}[\theta] & 0 & -1 \end{pmatrix}$$

This is the matrix in the next equation to (19.134):

```
{{(Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, -(1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ], η, 0},
  {-(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, 0, η},
  {-η, 0, (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, (1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ]},
  {0, -η, (1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2}} // MatrixForm
```

$$\begin{pmatrix} -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 + \text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}(1+\eta^2)\,\text{Sin}[\theta] & \eta & 0 \\ -\frac{1}{2}e^{i\phi}(1+\eta^2)\,\text{Sin}[\theta] & \text{Cos}\left[\frac{\theta}{2}\right]^2 - \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & 0 & \eta \\ -\eta & 0 & \text{Cos}\left[\frac{\theta}{2}\right]^2 - \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}(1+\eta^2)\,\text{Sin}[\theta] \\ 0 & -\eta & \frac{1}{2}e^{i\phi}(1+\eta^2)\,\text{Sin}[\theta] & -\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 + \text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

This is multiplication of the last two matrices:

```
{{η^2, 0, -η*Cos[θ], -η*Sin[θ]*Exp[-I*ϕ]}, {0, η^2, -η*Sin[θ]*Exp[I*ϕ], η*Cos[θ]},
   {η*Cos[θ], η*Sin[θ]*Exp[-I*ϕ], -1, 0}, {η*Sin[θ]*Exp[I*ϕ], -η*Cos[θ], 0, -1}}.
  {{(Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, -(1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ], η, 0},
   {-(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, 0, η},
   {-η, 0, (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, (1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ]},
   {0, -η, (1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2}};
```

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} -\frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \eta\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ -\frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

Now we compare this result with (19.135):

$$\begin{pmatrix} -\frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \eta\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ -\frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix} -$$

$$\left(1-\eta^2\right)*\begin{pmatrix} \eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\,\text{Sin}[\theta] & -\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the matrix identity under consideration is verified.

Determinant (19.136):

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix} \text{// MatrixForm}$$

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & -e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & -e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] & e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] & e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

```
Det[%];
```

```
FullSimplify[%]
```

$$\left(-1+\eta^2\right)^2$$

This is (19.136).

Finally, let us construct the tensor product in the l.h.s. of (19.125). This is our bi-spinor (19.131):

```
{η * Cos[θ / 2] * Exp[-(I * ϕ) / 2], η * Sin[θ / 2] * Exp[(I * ϕ) / 2],
  Cos[θ / 2] * Exp[-(I * ϕ) / 2], Sin[θ / 2] * Exp[(I * ϕ) / 2]} // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is its complex conjugate:

```
% /. ϕ → -ϕ // MatrixForm
```

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is the Dirac conjugate:

```
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}.{η * Cos[θ / 2] * Exp[(I * ϕ) / 2],
  η * Sin[θ / 2] * Exp[-(I * ϕ) / 2], Cos[θ / 2] * Exp[(I * ϕ) / 2], Sin[θ / 2] * Exp[-(I * ϕ) / 2]}
```

$$\left\{e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right],\ e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right],\ -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right],\ -e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right]\right\}$$

```
% // MatrixForm
```

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\eta\,\text{Cos}\left[\frac{\theta}{2}\right] \\ e^{-\frac{i\phi}{2}}\eta\,\text{Sin}\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\text{Cos}\left[\frac{\theta}{2}\right] \\ -e^{-\frac{i\phi}{2}}\text{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is their tensor product (19.135):

```
TensorProduct[{η * Cos[θ / 2] * Exp[-(I * ϕ) / 2],
    η * Sin[θ / 2] * Exp[(I * ϕ) / 2], Cos[θ / 2] * Exp[-(I * ϕ) / 2], Sin[θ / 2] * Exp[(I * ϕ) / 2]},
  {e^(I ϕ / 2) η Cos[θ / 2], e^(-I ϕ / 2) η Sin[θ / 2], -e^(I ϕ / 2) Cos[θ / 2], -e^(-I ϕ / 2) Sin[θ / 2]}] // MatrixForm
```

$$\begin{pmatrix} \eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & e^{-i\phi}\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -e^{-i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{i\phi}\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -e^{i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & e^{-i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\text{Cos}\left[\frac{\theta}{2}\right]^2 & -e^{-i\phi}\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -e^{i\phi}\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\eta^2\,(1+\text{Cos}[\theta]) & \frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] & -\frac{1}{2}\eta\,(1+\text{Cos}[\theta]) & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \frac{1}{2}\eta\,(-1+\text{Cos}[\theta]) \\ \frac{1}{2}\eta\,(1+\text{Cos}[\theta]) & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\,\text{Sin}[\theta] & \frac{1}{2}\,(-1+\text{Cos}[\theta]) \end{pmatrix}$$

This is our matrix in (19.135):

```
{{η^2 Cos[θ/2]^2, 1/2 η^2 E^(-I ϕ) Sin[θ], -η Cos[θ/2]^2, -1/2 E^(-I ϕ) η Sin[θ]},
 {1/2 η^2 E^(I ϕ) Sin[θ], η^2 Sin[θ/2]^2, -1/2 E^(I ϕ) η Sin[θ], -η Sin[θ/2]^2},
 {η Cos[θ/2]^2, 1/2 E^(-I ϕ) η Sin[θ], -Cos[θ/2]^2, -1/2 E^(-I ϕ) Sin[θ]},
 {1/2 E^(I ϕ) η Sin[θ], η Sin[θ/2]^2, -1/2 E^(I ϕ) Sin[θ], -Sin[θ/2]^2}} // MatrixForm
```

$$\begin{pmatrix} \eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 \\ \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\,\text{Sin}[\theta] & -\text{Sin}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

```
Simplify[% - %%] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Case E<0 and λ=1/2. The last identity (19.125):

This is the last tensor product of bi-spinors (19.132) in the polarization density matrix (19.125):

```
TensorProduct[{η*Sin[θ/2]*Exp[-(I*ϕ)/2], -η*Cos[θ/2]*Exp[(I*ϕ)/2], -Sin[θ/2]*Exp[-(I*ϕ)/2], Cos[θ/2]*Exp[(I*ϕ)/2]},
 {E^(I ϕ/2) η Sin[θ/2], -E^(-I ϕ/2) η Cos[θ/2], E^(I ϕ/2) Sin[θ/2], -E^(-I ϕ/2) Cos[θ/2]}] // MatrixForm
```

$$\begin{pmatrix} \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -e^{-i\phi}\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -e^{-i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] \\ -e^{i\phi}\eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -e^{i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \\ -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & e^{-i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\text{Sin}\left[\frac{\theta}{2}\right]^2 & e^{-i\phi}\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] \\ e^{i\phi}\eta\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & e^{i\phi}\,\text{Cos}\left[\frac{\theta}{2}\right]\text{Sin}\left[\frac{\theta}{2}\right] & -\text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

```
% /. Cos[θ/2] Sin[θ/2] -> 1/2 * Sin[θ] // MatrixForm
```

$$\begin{pmatrix} \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & \eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \\ -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{i\phi}\,\text{Sin}[\theta] & -\text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

Here, the bi-spinor (19.132):

```
{η * Sin[θ / 2] * Exp[-(I * ϕ) / 2], -η * Cos[θ / 2] * Exp[(I * ϕ) / 2],
  -Sin[θ / 2] * Exp[-(I * ϕ) / 2], Cos[θ / 2] * Exp[(I * ϕ) / 2]} // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{i\phi}{2}}\eta \sin\left[\frac{\theta}{2}\right] \\ -e^{\frac{i\phi}{2}}\eta \cos\left[\frac{\theta}{2}\right] \\ -e^{-\frac{i\phi}{2}} \sin\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}} \cos\left[\frac{\theta}{2}\right] \end{pmatrix}$$

and its Dirac conjugate in the above tensor product:

```
{e^(I ϕ / 2) η Sin[θ / 2], -e^(-I ϕ / 2) η Cos[θ / 2], e^(I ϕ / 2) Sin[θ / 2], -e^(-I ϕ / 2) Cos[θ / 2]} // MatrixForm
```

$$\begin{pmatrix} e^{\frac{i\phi}{2}}\eta \sin\left[\frac{\theta}{2}\right] \\ -e^{-\frac{i\phi}{2}}\eta \cos\left[\frac{\theta}{2}\right] \\ e^{\frac{i\phi}{2}} \sin\left[\frac{\theta}{2}\right] \\ -e^{-\frac{i\phi}{2}} \cos\left[\frac{\theta}{2}\right] \end{pmatrix}$$

This is the matrix in (19.134), once again:

```
{{η^2, 0, -η * Cos[θ], -η * Sin[θ] * Exp[-I * ϕ]},
  {0, η^2, -η * Sin[θ] * Exp[I * ϕ], η * Cos[θ]}, {η * Cos[θ], η * Sin[θ] * Exp[-I * ϕ], -1, 0},
  {η * Sin[θ] * Exp[I * ϕ], -η * Cos[θ], 0, -1}} // MatrixForm
```

$$\begin{pmatrix} \eta^2 & 0 & -\eta\cos[\theta] & -e^{-i\phi}\eta\sin[\theta] \\ 0 & \eta^2 & -e^{i\phi}\eta\sin[\theta] & \eta\cos[\theta] \\ \eta\cos[\theta] & e^{-i\phi}\eta\sin[\theta] & -1 & 0 \\ e^{i\phi}\eta\sin[\theta] & -\eta\cos[\theta] & 0 & -1 \end{pmatrix}$$

This is the matrix in the equation that is the next to (19.129), once again:

```
{{(Cos[θ / 2]) ^2 - η^2 * (Sin[θ / 2]) ^2, (1 / 2) * (1 + η^2) * Sin[θ] * Exp[-I * ϕ], -η, 0},
  {(1 / 2) * (1 + η^2) * Sin[θ] * Exp[I * ϕ], (Sin[θ / 2]) ^2 - η^2 * (Cos[θ / 2]) ^2, 0, -η},
  {η, 0, (Sin[θ / 2]) ^2 - η^2 * (Cos[θ / 2]) ^2, -(1 / 2) * (1 + η^2) * Sin[θ] * Exp[-I * ϕ]},
  {0, η, -(1 / 2) * (1 + η^2) * Sin[θ] * Exp[I * ϕ], (Cos[θ / 2]) ^2 - η^2 * (Sin[θ / 2]) ^2}} // MatrixForm
```

$$\begin{pmatrix} \cos\left[\frac{\theta}{2}\right]^2 - \eta^2 \sin\left[\frac{\theta}{2}\right]^2 & \frac{1}{2} e^{-i\phi}\left(1+\eta^2\right)\sin[\theta] & -\eta & 0 \\ \frac{1}{2} e^{i\phi}\left(1+\eta^2\right)\sin[\theta] & -\eta^2\cos\left[\frac{\theta}{2}\right]^2 + \sin\left[\frac{\theta}{2}\right]^2 & 0 & -\eta \\ \eta & 0 & -\eta^2\cos\left[\frac{\theta}{2}\right]^2 + \sin\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2} e^{-i\phi}\left(1+\eta^2\right)\sin[\theta] \\ 0 & \eta & -\frac{1}{2} e^{i\phi}\left(1+\eta^2\right)\sin[\theta] & \cos\left[\frac{\theta}{2}\right]^2 - \eta^2 \sin\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

The multiplication of the last two matrices results in:

```
{{η^2, 0, -η*Cos[θ], -η*Sin[θ]*Exp[-I*ϕ]}, {0, η^2, -η*Sin[θ]*Exp[I*ϕ], η*Cos[θ]},
  {η*Cos[θ], η*Sin[θ]*Exp[-I*ϕ], -1, 0}, {η*Sin[θ]*Exp[I*ϕ], -η*Cos[θ], 0, -1}}.
 {{(Cos[θ/2])^2 - η^2*(Sin[θ/2])^2, (1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ], -η, 0},
  {(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, 0, -η},
  {η, 0, (Sin[θ/2])^2 - η^2*(Cos[θ/2])^2, -(1/2)*(1+η^2)*Sin[θ]*Exp[-I*ϕ]},
  {0, η, -(1/2)*(1+η^2)*Sin[θ]*Exp[I*ϕ], (Cos[θ/2])^2 - η^2*(Sin[θ/2])^2}};
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & -\frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & -\frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) \\ \eta\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) \end{pmatrix}$$

Finally, we compare the results:

$$\begin{pmatrix} \frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(-1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta^2\left(-1+\eta^2\right)\text{Sin}[\theta] & -\frac{1}{2}\eta^2\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & \frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & -\frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) \\ \eta\left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \left(-1+\eta^2\right)\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\eta\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) & -\frac{1}{2}e^{i\phi}\left(-1+\eta^2\right)\text{Sin}[\theta] & \frac{1}{2}\left(-1+\eta^2\right)\left(1+\text{Cos}[\theta]\right) \end{pmatrix} -$$

$$\left(1-\eta^2\right)*\begin{pmatrix} \eta^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta^2\,\text{Sin}[\theta] & \eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] \\ -\frac{1}{2}e^{i\phi}\eta^2\,\text{Sin}[\theta] & \eta^2\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & -\frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & \eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 \\ -\eta\,\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\eta\,\text{Sin}[\theta] & -\text{Sin}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{-i\phi}\,\text{Sin}[\theta] \\ \frac{1}{2}e^{i\phi}\eta\,\text{Sin}[\theta] & -\eta\,\text{Cos}\left[\frac{\theta}{2}\right]^2 & \frac{1}{2}e^{i\phi}\,\text{Sin}[\theta] & -\text{Cos}\left[\frac{\theta}{2}\right]^2 \end{pmatrix};$$

```
FullSimplify[%] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, all cases of (19.124)-(19.125) are verified by Mathematica.

Appendix: Matrix Multiplication in Mathematica

Matrix multiplication

```
A = {{a11, a12}, {a21, a22}}
```

```
{{a11, a12}, {a21, a22}}
```

```
A // MatrixForm
```

$$\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}$$

```
B = {{b11, b12}, {b21, b22}}
```

```
{{b11, b12}, {b21, b22}}
```

```
B // MatrixForm
```

$$\begin{pmatrix} b11 & b12 \\ b21 & b22 \end{pmatrix}$$

```
{{a11, a12}, {a21, a22}}.{{b11, b12}, {b21, b22}} // MatrixForm
```

$$\begin{pmatrix} a11\,b11 + a12\,b21 & a11\,b12 + a12\,b22 \\ a21\,b11 + a22\,b21 & a21\,b12 + a22\,b22 \end{pmatrix}$$

```
A.B // MatrixForm
```

$$\begin{pmatrix} a11\,b11 + a12\,b21 & a11\,b12 + a12\,b22 \\ a21\,b11 + a22\,b21 & a21\,b12 + a22\,b22 \end{pmatrix}$$

Pauli matrices

```
Clear[{σ1, σ2, σ3}]
```

Clear: {σ1, σ2, σ3} is not a symbol or a string.

```
σ1 = {{0, 1}, {1, 0}} // MatrixForm
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

```
σ2 = {{0, -I}, {I, 0}} // MatrixForm
```

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

```
σ3 = {{1, 0}, {0, -1}} // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
σ1.σ2
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

```
Simplify[%]
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

```
{{0, 1}, {1, 0}}.{{0, -I}, {I, 0}} // MatrixForm
```

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

```
σ1.σ3
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$$

```
{{0, -1}, {1, 0}}
```

```
{{0, 1}, {1, 0}}.{{1, 0}, {0, -1}} // MatrixForm
```

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

```
σ2.σ3
```

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
{{0, -I}, {I, 0}}.{{1, 0}, {0, -1}} // MatrixForm
```

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Tensor product

```
TensorProduct[{a, b, c}, {α, β, γ}] // MatrixForm
```

$$\begin{pmatrix} a\,\alpha & a\,\beta & a\,\gamma \\ b\,\alpha & b\,\beta & b\,\gamma \\ c\,\alpha & c\,\beta & c\,\gamma \end{pmatrix}$$

```
TensorProduct[{1, 2, 3}, {3, 1, 2}] // MatrixForm
```

$$\begin{pmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 9 & 3 & 6 \end{pmatrix}$$

```
v1 = {a, b, c} // MatrixForm
```

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

```
v2 = {α, β, γ} // MatrixForm
```

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

```
TensorProduct[v1, v2] // MatrixForm
```

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

```
TensorProduct[{d}, {{{a α, a β, a γ}, {b α, b β, b γ}, {c α, c β, c γ}}}] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} a\,d\,\alpha & a\,d\,\beta & a\,d\,\gamma \\ b\,d\,\alpha & b\,d\,\beta & b\,d\,\gamma \\ c\,d\,\alpha & c\,d\,\beta & c\,d\,\gamma \end{pmatrix} \end{pmatrix}$$

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Matrix Approach to Helicity States of Dirac Free Particles
by Sergei Suslov

Wolfram Community, STAFF PICKS, June 9, 2023
https://community.wolfram.com/groups/-/m/t/2933767

[Here, we present a revised version to match Chapter 19 in the proceedings under consideration.]