Chapter 18

Thermodynamics of the dynamical Casimir effect

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18.1 Introduction

Perhaps one of the most counterintuitive results offered by quantum theory refers to the appreciation that the vacuum cannot be understood simply as an empty place completely devoid of content, but rather as a dynamical entity endowed with a rich structure and a fluctuating character [1, 2]. This is best exemplified by the *Casimir effect* [3], which predicts the emergence of attractive forces between a pair of parallel ideal mirrors in vacuum due to the changes in the zero-point energy of the quantum electromagnetic field. Another surprising prediction in this vein is contained in the statement of the non-stationary counterpart of the last phenomenon: the *dynamical Casimir effect* (DCE), which asserts that rapidly changing external properties of a quantum field —such as the time-dependent boundary conditions imposed by moving mirrors– can induce the process of creation of pairs of particles from the vacuum. Firstly proposed by Gerald Moore [4] and subsequently explored by DeWitt [5] and Fulling and Davies [6, 7] during the 1970s, the research program of DCE has received numerous developments in more than fifty years of existence, encompassing the considerations from semi-transparent mirrors [8–13], the quantized motion of the walls [14–20], effective Hamiltonian descriptions [14, 21–25], distinct geometrical configurations [26–30], gravitational field effects [31, 32], nonlinear interactions [33–35] and entanglement dynamics [36, 37]. For a comprehensive overview, interested readers are directed to the recent review [38].

As widely acknowledged in the literature, the DCE can only occur if the quantum field under consideration is subjected to *non-adiabatic* changes of one of its external parameters. This result strongly suggests that once particles are created from the vacuum, one can not expect to reverse the process in a spontaneous manner, ultimately meaning that the effect must exhibit an intrinsic irreversible character. Despite this intuition, the study of the irreversible dynamics in the DCE is still unexplored to this date, begging the question of how one can confirm such a diagnosis and, if positive, what are the sources possibly playing a role in the generation of irreversibility. It turns out that thermodynamics already contains, in its own formalism, a physical quantity that exactly measures irreversibility: the entropy of the system. In this sense, a process is said to be irreversible if the variation of its entropy is an increasing function of time. Therefore, by focusing on a physical configuration where a quantum field is constrained by classical moving mirrors, we are then immediately led to consider the study of irreversibility in terms of the entropy production in isolated quantum systems, where the time evolution follows a unitary process.

In this context, although a natural choice, the von Neumman entropy cannot possibly be a viable alternative to a microscopic definition of entropy since it is conserved for unitary transformation and, therefore, contradicts the second law of thermodynamics for isolated systems. To surpass this fundamental issue, we choose as the thermodynamic entropy for our system a quantity introduced in Ref. [39] under the name of *diagonal entropy* as the microscopic entropy for closed systems. Such entropy is completely defined in terms of the diagonal elements of the system's density operator in the energy eigenbasis, implying therefore, its increase for any process, whether unitary or not, that induces transitions in the energy eigenbasis. Only when the system's Hamiltonian changes slowly enough will the diagonal entropy remain unchanged. This aligns with our intuition based on the classical definition of thermodynamic entropy, which does not increase for quasi-static processes [40, 41].

In regard to the initial question of how to explore the irreversible dynamics of the DCE, we choose to model the effect in terms of the most simple (but non-trivial) system that reproduces the phenomenon of particle creation from the vacuum: a quantum scalar field confined in a one-dimensional cavity by a pair of ideal mirrors, with one fixed in a given position and the other allowed to move in a prescribed trajectory. With this configuration in mind the central goal of this presentation is therefore by addressing the question of how much thermodynamic entropy is produced in our field if one takes into account the correspondent time-dependent nature of the boundary conditions. We provide answers to this question through two distinct approaches. In a first moment we employ an effective Hamiltonian method based on Ref. [22] to compute the global entropy of the field (for the entirety of the field modes) within the short-time regime. Using such theory we demonstrate that the entropy production of the system is intrinsically related to the generation of quantum coherences in the instantaneous energy levels, in agreement with the gauge theory developed in Ref. [43]. Specifically to the second part of this presentation we opt to a distinct approach to study the long-time field dynamics which allow us to calculate the reduced diagonal entropy for a distinct field mode. The surprising results is that the dynamics of the entropy is governed by the generation of entanglement between the select mode and the rest of the field modes. Both approaches, which are distinct although complementary, allow us to link the irreversibility of the field dynamics with two fundamental quantum features: quantum coherence and entanglement.

18.2 Thermodynamics of isolated quantum systems

Even though the fundamental laws of physics are said to be time-symmetric —not distinguishing a preferred direction in time— we recurrently encounter phenomena in our daily live, termed as irreversible, that seems to manifest spontaneously in only one way. Thermodynamics was perhaps the first physical theory to incorporate in its formalism the distinction between such phenomena with the introduction of entropy as a measure of irreversibility. In it, the ubiquitous presence of irreversible processes in the macroscopic world is explained by the second law of thermodynamics, which asserts that the total thermodynamic entropy of a closed systems can never decrease in time [44]. With the establishment of statistical considerations in the foundations of physics, the nature of the second law could finally be elucidated not as fundamental law of nature, but as a statistical statement that holds true on the overwhelmingly number of cases. Indeed, when deviations from the second law are relevant, stronger principles known as fluctuation theorems arise [45, 46], and irreversibility is updated in terms of those processes that tends to increase entropy on average.

In recent decades physics has witnessed a massive technological development in experimental techniques, enabling the manipulation over the physical state of quantum systems with a precision never seem before. In this context, experimental physicists and engineers can now investigate in a day to day basis the dynamics of small quantum systems in contact with thermal baths, as well as isolated many-body systems with a high degree of tunability (for instance, the experiments with ultra cold atoms [48, 49]). The immediate consequence of such considerations is that experimentalists have now an ever-increasing control over the behaviour of quantum processes even out of equilibrium. In this sense, although the status of thermodynamics as a successful phenomenological theory remain intact, immediate difficulties arise when attempting to apply it to non-equilibrium situations, as mentioned above. As a result, the duty of comprehending how thermodynamics emerges from quantum mechanical and statistical considerations —frequently known as *quantum thermodynamics*– is of increasing significance. Over the years various routes have come to light in the quest of grasping thermodynamics from a microscopic level. Noteworthy developments include those based upon statistical physics [45, 50], resource theories [51], density functional theories [52], axiomatic formulations [53] and information theories [54, 55]. For a broad examination of entropy production in both classical and quantum settings, we suggest Ref. [56] along its associated references.

A special conundrum in our pretension to explore the entropy production in the DCE refers to the problem of how to establish the correct definition of microscopic entropy that naturally extends the laws of thermodynamics to the context of quantum mechanics both for open and isolated systems [39, 42, 57, 58]. A popular choice to the position of microscopic entropy is occupied by the von Neumann entropy S_{vN}, defined for a system's state $\hat{\rho}$ as

$$S_{vN}(\hat{\rho}) = -\mathrm{Tr}\left\{\hat{\rho}\ln\hat{\rho}\right\}. \tag{18.1}$$

Such popularity occurs mainly due to its tremendous success in applications to information and communication theories. Mostly important, the von Neumann entropy is consistent with thermodynamic considerations when the chosen system is characterised by an statistical ensemble, either *in* or *out of equilibrium.* This is justified in light of the extensive use of S_{vN} in quantum thermodynamics to study small quantum systems in weak contact with a large thermal bath.

Nevertheless, there exist strong reasons against the understanding that the von Neumman entropy correspond to the thermodynamics entropy *in general.* This can be seen more explicitly with the statement of its invariance under unitary evolution, as in

$$S_{\mathrm{vN}}(\hat{\rho}) = S_{\mathrm{vN}}(\hat{U}\hat{\rho}\hat{U}^{\dagger}), \tag{18.2}$$

where $\hat{U}$ is the time evolution operator that generate the dynamics of $\hat{\rho}$. Consequently the interpretation of S_{vN} as the microscopic definition of entropy entail the prediction that the thermodynamic entropy for isolated system must be conserved for any process, which is in complete contradiction with the empirical observation during the free expansion of a gas or in the mixing of liquids [42]. Another important postulate violated by S_{vN} refers to the uniqueness of entropy as a function of the system's energy. This violation occur specifically in isolated systems, where the change in S_{vN} must vanish in any process despite some possible non-zero change for the system's energy [39]. In this respect, although the von Neumman entropy stands as an essential informational tool and coincides with the thermodynamic entropy for important situations, we can not elevate S_{vN} to the status of microscopic entropy in general.

18.2.1 Diagonal entropy

Another contestant for microscopic entropy —first introduced in Ref. [39] as the thermodynamic entropy for closed systems— is the so called *diagonal entropy* or *d-entropy* S_d, defined as

$$S_d := -\sum_n \rho_{nn} \ln \rho_{nn}, \tag{18.3}$$

where ρ_{nn} are the diagonal components of the system's density operator in the eigenbasis of the Hamiltonian (energy eigenstates). An important point to mention is the ambiguity in S_d expression when the energy eigenstates are degenerate. Since such circumstance generally occur accidentally or as a result of system symmetries, they are usually absent and will be ignored throughout the text.

Such entropy can be understood as quantifying the randomness observed within the energy eigenbasis. This is evident in the commonly encounter situation where the system has a sufficiently large dimensionality and quantum state tomography becomes impractical [59]. For such physical setting it is reasonable to assume we have access only to energy measurement of the quantum system. Therefore, for a general process (unitary or not) it become inevitable the emergence of transitions between instantaneous energy levels and the consequent development of quantum coherence and entanglement among different parts of the system. The diagonal entropy can then be identified as measuring the loss of information due to the limited set of measurements available.

It is also important to highlight that a very close related quantity, now known as observational entropy, can be defined as a coarse-grained variant of the diagonal entropy [42]. Hence, the results obtained here also hold within the context of observational entropy. Furthermore, an emerging notion of entropy which corresponds precisely to the S_d can be derived from a recent framework for quantum thermodynamics introduced in Ref. [43] where physical quantities that are invariant under an emergent gauge group. Such correspondence underscores the result that gauge-invariant definition of heat is intricately linked with transitions between energy levels [43].

As discussed in Ref. [39], there are two main reasons to support the claim that only the diagonal contributions of the density operator are essential for the thermodynamic description of quantum systems. The first one is contained in the characterization of a sufficiently complex system that have achieved a steady state after some process that occur in the distant past, as in

$$\hat{\rho}(t) = \sum_{mn} \rho_{nm} e^{-i(E_m - E_n)t} \left| m \right\rangle \left\langle n \right| , \tag{18.4}$$

with $\hat{H} \left| n \right\rangle = E_n \left| n \right\rangle$ being the stationary Schrodinger equation for the final Hamiltonian (without time-dependence). By anticipating that even isolated quantum systems are likely to respect some form o ergodic behaviour, one expect to connect time averages of a arbitrary thermodynamic observable $\hat{O}$ with its equilibrium ensemble average, as

$$\overline{\langle \hat{O}(t) \rangle} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathrm{dt} \langle \hat{\mathrm{O}}(\mathrm{t}) \rangle, \tag{18.5}$$

where $\langle \hat{O}(t) \rangle = \mathrm{Tr} \left\{ \hat{\rho}(\mathrm{t}) \hat{\mathrm{O}} \right\}$. In terms of the steady state (18.4), we obtain

$$\overline{\langle \hat{O}(t) \rangle} = \sum_{nm} \rho_{nm} O_{nm} \overline{\langle e^{-i(E_m - E_n)t} \rangle} = \sum_{nm} \rho_{nn} O_{nn}, \tag{18.6}$$

where $O_{nm} \equiv \left\langle n \right| \hat{O} \left| m \right\rangle$ are the observable matrix elements in the non-degenerate energy eigenstate basis. Since expression (18.6) is only dependent on the time-independent diagonal elements of $\hat{\rho}$, we can the conclude that relevant information for thermodynamic measurements (associated with final steady states) is only present in the diagonal terms ρ_{nn}.

A second argument putted forward by Ref [39] uses the fact that slowly varying Hamiltonians cannot induce transitions between instantaneous eigenstates, a well-known result contained in the *adiabatic theorem for quantum mechanics* [61]. Being more specific, if one consider a time-dependent Hamiltonian $\hat{H}(t)$ respecting $\hat{H}(t) \left| n(t) \right\rangle = E_n(t) \left| n(t) \right\rangle$ and a solution of the time-dependent Schrödinger equation given by the state

$$\left| \Psi(t) \right\rangle = \sum_n c_n(t) \left| n(t) \right\rangle , \tag{18.7}$$

if $\hat{H}(t)$ changes sufficiently slow in time, *i.e.*, a quasi-static process, it is possible to show that the time-dependent amplitudes $c_n(t)$ must evolves in time through [62]

$$c_n(t) = c_n(0) \exp -i \int_0^t \left[E_m(t') - i \left\langle m(t') \right| \frac{\partial}{\partial t'} \left| m(t') \right\rangle \right] \mathrm{dt}'. \tag{18.8}$$

The first consequence of expression (18.8) is the realizations that the value for the eigenstate probabilities $|c_n(t)|^2 = |c_n(0)|^2$ remain constant over time. This means that if the system start at $t = 0$ as an eigenstate of the initial Hamiltonian $\hat{H}(0)$, then will the passage of time one expects the system to remain as an eigenstate of the instantaneous Hamiltonian $\hat{H}(t)$ during all time (only differing by a time-dependent phase factor). As a consequence, the instantaneous diagonal components of the density operator, $\rho_{nn}(t) = |c_n(t)|^2$, are conserved over time (for quasi-static processes). Therefore, any entropic quantity only dependent on ρ_{nn}, such as S_d, should be invariant under adiabatic process, as it is expected from a thermodynamic entropy.

Moreover, the diagonal entropy satisfies all the key properties expected from the desired thermodynamic entropy. In the case of equilibrium states (where $\hat{\rho}$ is stationary and therefore, diagonal), the d-entropy ends up being identical to the von Neumman entropy. Consequently, S_d must exhibit important properties such as extensively, positivity and vanishing under the zero-temperature limit.

For more general settings, as when S_d changes in time, if the initial state is stationary one can show

$$S_d(t) \geq S_{vN}(0), \tag{18.9}$$

for any time-dependent process in an isolated system that [39]. We remark that Eq. (18.9) does not imply that the diagonal entropy is always a monotonic function of time. To better grasp the last expression, we analyse its increase in terms of the evolved density operator

$$\hat{\rho}(t) = \hat{U}^{\dagger}(t)\hat{\rho}(0)\hat{U}(t), \tag{18.10}$$

with $\hat{U}(t)$ being the time-evolution operator. Considering an initial diagonal state $\hat{\rho}(0) = \sum_m \rho_{mm}(0) \left|m\right\rangle \left\langle m\right|$, the diagonal part of the evolved state can be shown so satisfy

$$\rho_{nn}(t) = \left\langle n\right| \hat{\rho}(t) \left|n\right\rangle = \sum_m P_{nm}\rho_{mm}(0), \tag{18.11}$$

where $P_{nm} = |U_{mn}|^2$ represents the transition rates among the instantaneous energy eigenvalues. Given the doubly stochastic nature of the matrix P ($\sum_m P_{mn} = \sum_n P_{nm} = 1$), one expect the existence of a tendency for the system to promote a uniform spreading of probability density ρ_{nn} among the energy spectrum. Consequently, as S_d quantifies the spread of ρ_{nn}, an arbitrary dynamical process must necessarily either increases or remains constant the value for the diagonal entropy.

Furthermore, another indispensable property we should expect any thermodynamic entropy expression to respect, refers to the fundamental thermodynamic relation

$$\Delta U = T\Delta S - \sum_j \frac{\partial U}{\partial \lambda_i}\bigg|_S \Delta\lambda_i, \tag{18.12}$$

which emerges from the postulate in thermodynamics of requiring the system's entropy to be an unique function of energy E and external parameters λ_1, λ_2, etc.

The change ΔU of the system's energy $U(t) = \sum_n \rho_{nn}(t)E_n(t)$ in terms of a liner order expansion on $\Delta\rho$ and $\Delta\lambda$ can be easily obtained in the form

$$\Delta U \approx \sum_n \Delta E_n(t)\rho_{nn}(0) + \sum_n E_n(0)\Delta\rho_{nn}(t), \tag{18.13}$$

where $\Delta E_n(t) = E_n(t) - E_n(0)$ is the change in the instantaneous energy eigenstates due to the time evolution while $\Delta\rho_{nn}(t) = \rho_{nn}(t) - \rho_{nn}(0)$ is the change in the diagonal components of $\hat{\rho}$.

In the adiabatic limit, where there is no transitions between instantaneous energy levels ($\Delta\rho_{nn}(t) = 0$), the first term in Eq. (18.13) given by $\Delta E_{\mathrm{ad}} = \sum_n \Delta E_n(t)\rho_{nn}(0)$, can be identified as the adiabatic change of the system's energy, i.e., the work done by the system due to the changes of the external parameter λ_i. As a consequence, the second term $Q(t) = \sum_n E_n(0)\Delta\rho_{nn}(t)$ of Eq. (18.13) is then represented by the non-adiabatic change of the energy i.e., the heat generated by the system under a non-quasi-static process.

Given $\sum_n \Delta\rho_{nn}(t) = 0$, a change in the diagonal entropy to the leading order in $\Delta\hat{\rho}_{nn}(t)$ takes the following form

$$\Delta S_d \approx -\sum_n \left[\Delta\rho_{nn}(t)\ln\rho_{nn}(0) + \rho_{nn}(0)\Delta\ln\rho_{nn}(t)\right] = -\sum_n \Delta\rho_{nn}(t)\ln\rho_{nn}(0). \quad (18.14)$$

Considering an initial thermal state

$$\hat{\rho}(0) = \sum_n \frac{1}{Z} e^{-E_n(0)/T} \left|n\right\rangle\left\langle n\right|, \qquad \text{with} \quad Z = \sum_m e^{-E_m|(0)/T}, \quad (18.15)$$

an immediate consequence of Eq. (18.14) is that $\Delta S_d = \sum_n \Delta\rho_{nn}(t)E_n(0)/T = Q(t)/T$. Using such identity, one can rewrite Eq. (18.13) as

$$\Delta U \approx \Delta E_{\text{ad}} + T\Delta S_d. \quad (18.16)$$

As ΔE_{ad} represents the work done by the system and can be written in terms of the partial derivatives of the system energy over the external parameters, as in

$$\Delta E_{\text{ad}} = \sum_j \frac{\partial U}{\partial\lambda_i}\bigg|_{S_d} \Delta\lambda_i, \quad (18.17)$$

which makes Eq. (18.16) equivalent to the fundamental thermodynamic relation (18.12).

Finally, the diagonal entropy can also be shown to be additive in thermal equilibrium due to the correspondence between S_d and S_{vN} for such regime. In non-equilibrium conditions, however, the situation is more subtle. Although the sum of thermodynamic entropies for subsystems is equal to the sum of their diagonal entropies, their sum is not equal to the total S_d. Moreover, another interesting result refer to the case of two initially uncoupled systems each in local equilibrium. If such systems are allowed to to interact, the sum of their diagonal entropy should satisfy the inequality $S_d^{(1)}(t)+S_d^{(2)}(t) \geq S_d^{(1)}(0) + S_d^{(2)}(0)$. This result aligns with the prediction from thermodynamics, as the second law demands the sum of entropies to increase or remain constant.

18.3 Dynamical Casimir effect

In order to discuss the dynamical Casimir effect we begin considering a one-dimensional ideal cavity whose mirrors are located at positions $x = 0$ and $L(t)$, with $L(t)$ being an externally prescribed trajectory. Confined in this cavity, we have a massless real scalar field $\Phi(x,t)$ satisfying the wave equation

$$\left(\partial_t^2 - \partial_x^2\right)\Phi(x,t) = 0 \quad \text{with} \quad \Phi(0,t) = \Phi(L(t),t) = 0, \quad (18.18)$$

where the non-stationary boundary conditions imposed on the field characterises the ideal nature of the mirrors (perfect reflectors).

The quantum description for the system is obtained with canonical quantization procedure, where the classical field $\Phi(x,t)$ and its momentum conjugated $\Pi(x,t) = \partial_t\Phi(x,t)$

are substituted by operators $\hat{\Phi}(x,t)$ and $\hat{\Pi}(x,t)$ satisfying the equal-time commutation relations

$$\left[\hat{\Phi}(x,t),\hat{\Pi}(x',t)\right] = i\delta(x-x') \quad \text{and} \quad \left[\hat{\Phi}(x,t),\hat{\Phi}(x',t)\right] = \left[\hat{\Pi}(x,t),\hat{\Pi}(x',t)\right] = 0.$$

In general when a quantum field is under the influence of time-dependent backgrounds —as in the case of the non-trivial boundary conditions (18.18)— one cannot expect the notion of particle for the theory to be always well-defined. For this reason we will divide the problem in three stages, with an initial and final interval of time in which the cavity is static and the definition of particles is meaningful. More specifically, we consider the cavity to begin at rest (initial interval *in*: $t \leq 0$) with with constant size $L(t<0) = L_0$. Subsequently, the second mirror is then allow to move in its prescribed trajectory L(t) during the finite interval $0 < t < T$, while at the final interval *out*: $t \geq T$, the cavity then returns to its static configuration with constant size L_T.

To account for the dynamical nature conditions (18.18) we utilise the method known as the *instantaneous basis approach.* The main feature of this approach revolves around the introduction of a set of complete and orthonormal basis functions

$$\varphi_k(x,t) = \sqrt{\frac{2}{L(t)}}\sin\left(\omega_k(t)x\right), \quad \text{with} \quad \omega_k(t) = \frac{k\pi}{L(t)} \quad (k = 1,2,\dots) \tag{18.19}$$

which is an eigenfunction of the one-dimensional Laplacian operator $-\partial_x^2$ with the dynamical boundary condition (18.18). With this at hands, the quantum fields $\hat{\Phi}(x,t)$ and $\hat{\Pi}(x,t)$ can be expanded instantaneously as

$$\hat{\Phi}(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k(t)}}\left[\hat{a}_k(t)e^{-i\Omega_k(t)} + \hat{a}_k^\dagger(t)e^{i\Omega_k(t)}\right]\varphi_k(x,t) \tag{18.20a}$$

$$\hat{\Pi}(x,t) = i\sum_k \sqrt{\frac{\omega_k(t)}{2}}\left[\hat{a}_k^\dagger(t)e^{i\Omega_k(t)} - \hat{a}_k(t)e^{-i\Omega_k(t)}\right]\varphi_k(x,t), \tag{18.20b}$$

where $\Omega_k(t) = \int_0^t dt'\omega_k(t')$ and the *instantaneous* annihilation and creation operators $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ satisfy the standard equal times commutation relations

$$\left[\hat{a}_k(t),\hat{a}_k^\dagger(t)\right] = \delta_{kj} \quad \text{and} \quad \left[\hat{a}_k(t),\hat{a}_k(t)\right] = \left[\hat{a}_k^\dagger(t),\hat{a}_k^\dagger(t)\right] = 0.$$

Here the name instantaneous refers to the physical interpretation that if we "freeze" the system at some instant T, then $\hat{a}_k(t_0)$ and $\hat{a}_k^\dagger(t_0)$ must describe the operators as if the cavity have been stopped at a fixed size $l(t_0)$ (where the notion of particle is well-defined) [22].

Another important use for the creation and annihilation operators is in order to construct a particular basis for the Hilbert space of the theory. In this respect, the *instantaneous* vacuum state $|0;t_0\rangle$ is defined as the state annihilated by all $\hat{a}_k(t_0)$, whereas a general particle state can be constructed by the application of the creation operator $\hat{a}_k^\dagger(T)$ on this vacuum state

$$|\boldsymbol{n};t_0\rangle = |n_{k_1},n_{k_2},\dots;t_0\rangle = \prod_i \frac{1}{\sqrt{n_{k_i}!}}\left[\hat{a}_{k_i}^\dagger(t_0)\right]^{n_{k_i}}|0;t_0\rangle\,,$$

with n_{k_i} representing the number of particles in the k_i-th mode after the cavity has stopped at the instant t_0. From now one we will speak of the vacuum state defined at the interval of time *in* as $|0, in\rangle$.

Taking the time derivative of Eqs. (18.20) we obtain, after some algebra, the following set of differential equations for the annihilation operator

$$\dot{\hat{a}}_j(t) = \sum_k \left[A_{kj}(t)a_k(t) + B^*_{kj}(t)a^\dagger_k(t)\right], \tag{18.21a}$$

$$\dot{\hat{a}}^\dagger_j(t) = \sum_k \left[A^*_{kj}(t)a^\dagger_k(t) + B_{kj}(t)a_k(t)\right], \tag{18.21b}$$

In this equation, we defined the coefficients

$$\left.\begin{array}{c} A_{kj}(t) \\ B_{kj}(t) \end{array}\right\} = (-1)^{j\text{-}k}\frac{\sqrt{jk}}{j \mp k}\frac{\dot{L}(t)}{L(t)}e^{-i[\Omega_k(t) \mp \Omega_j(t)]}. \tag{18.22}$$

18.3.1 Particle creation process

To compute the number of particles created due to the DCE, we first need to relate the notion of particle given by the *instantaneous* operators $\hat{a}_k(t)$ and $\hat{a}^\dagger_k(t)$ with the initial operators $\hat{a}^{in}_k \equiv \hat{a}_k(t=0)$ and $\hat{a}^{in\dagger}_k \equiv \hat{a}^\dagger_k(t=0)$. Solutions for the set of differential equations (18.21) can be obtained through considerations of the Bogoliubov transformations

$$\hat{a}_k(t) = \sum_j \left[\alpha_{jk}(t)\hat{a}^{in}_j + \beta^*_{jk}(t)\hat{a}^{in\dagger}_j\right] \tag{18.23}$$

where $\alpha_{jk}(t)$ and $\beta_{jk}(t)$ are the "instantaneous" Bogoliubov coefficients with the initial conditions $\alpha_{jk}(0) = \delta_{jk}$ and $\beta_{jk}(0) = 0$. Inserting the expression (18.23) and its hermitian conjugated into Eq. (18.21a) and equating the last expression with the time derivative of Eq. (18.23), we find the following differential equations for the Bogoliubov coefficients

$$\dot{\alpha}_{jk}(t) = \sum_{\mathbf{j'}} \left[A_{kj}(t)\alpha_{j'j}(t) + B^*_{kj}(t)\beta_{j'j}(t)\right], \tag{18.24a}$$

$$\dot{\beta}_{jk}(t) = \sum_{\mathbf{j'}} \left[B_{kj}(t)\alpha_{j'j}(t) + A^*_{kj}(t)\beta_{j'j}(t)\right]. \tag{18.24b}$$

Since the coefficients $A_{kj}(t)$ and $B_{kj}(t)$ are of order $\mathcal{O}[\lambda(t)]$ for $\lambda(t) = \dot{L}(t)/L(t)$, if we suppose the cavity to move much slower than the speed of light ($\dot{L}(t) \ll 1$), one can expand the Bogoliubov coefficients in terms of $\lambda(t)$. In the first order one can calculate their expressions to be

$$\alpha_{jk}(t) = \delta_{jk} + \int_0^t \mathrm{dt'}\ \mathrm{A_{kj}(t')}, \quad \text{and} \quad \beta_{\mathrm{jk}}(\mathrm{t}) = \int_0^{\mathrm{t}} \mathrm{dt'}\ \mathrm{B_{kj}(t')}. \tag{18.25}$$

Therefore, the number of particles created after the cavity returns to a static configuration at instant of time T can be computed with the help of the Bogoliubov transformations

$$N(T) = \langle 0; in| \sum_k \hat{a}^\dagger_k(T)\hat{a}_k(T)\,|0; in\rangle = \sum_{kj} |\beta_{jk}(T)|^2 \tag{18.26}$$

In general, β_{jk} is non-zero when time-dependent boundary conditions are imposed on the field. This last equation characterizes the DCE as the quantum field phenomenon of particle creation from the vacuum due to the time-dependent nature of the imposed boundary conditions.

18.3.2 Effective Hamiltonian

To investigate the dynamics of the system's irreversibility one must obtain time evolution of the density operator $\hat{\rho}$. Identifying Eq. (18.21) as the Heisenberg equation of motion for the instantaneous operators, it is straightforward to write down the effective Hamiltonian for the theory by considering the most general quadratic operator

$$\begin{aligned} \hat{H}(t) \quad = \quad & \sum_{kl} \Big[\mathcal{A}_{km}(t)\hat{a}_k^\dagger(t)\hat{a}_m^\dagger(t) + \mathcal{B}_{km}(t)\hat{a}_k^\dagger(t)\hat{a}_m(t) \\ & + \mathcal{C}_{km}(t)\hat{a}_m^\dagger(t)\hat{a}_k(t) + \mathcal{D}_{km}(t)\hat{a}_k(t)\hat{a}_m(t) \Big] , \end{aligned} \tag{18.27}$$

which is: (i) hermitian, by satisfying the conditions $\mathcal{A}_{kl}(t) = \mathcal{D}^*_{km}(t)$, $\mathcal{B}_{kl}(t) = \mathcal{C}^*_{km}(t)$ and (ii) invariant over an index change, with the conditions $\mathcal{A}_{km}(t) = \mathcal{A}_{mk}(t)$, $\mathcal{D}_{km}(t) = \mathcal{D}_{mk}(t)$, $\mathcal{B}_{km}(t) = \mathcal{C}_{mk}(t)$ and $\mathcal{B}_{mk}(t) = \mathcal{C}_{km}(t)$.

Suppressing the notation for time dependence, the correspondent Heisenberg equation of motion for the annihilation and creation operators is therefore

$$\dot{\hat{a}}_j(t) = i\left[\hat{H}(t), \hat{a}_j(t)\right] = -i\sum_k \Big\{ \left[\mathcal{A}_{kj}(t) + \mathcal{A}_{jk}(t)\right]\hat{a}_k^\dagger(t) + \left[\mathcal{B}_{jk}(t) + \mathcal{C}_{kj}(t)\right]\hat{a}_k(t) \Big\} \tag{18.28}$$

$$\dot{\hat{a}}_j^\dagger(t) = i\left[\hat{H}(t), \hat{a}_j^\dagger(t)\right] = i\sum_k \Big\{ \left[\mathcal{D}_{kj}(t) + \mathcal{D}_{jk}(t)\right]\hat{a}_k(t) + \left[\mathcal{B}_{kj}(t) + \mathcal{C}_{jk}(t)\right]\hat{a}_k^\dagger(t) \Big\}. \tag{18.29}$$

Comparing (18.21a) with (18.28) and (18.21b) with (18.29), we obtain the following system

$$\begin{aligned} -i\left[\mathcal{A}_{kj}(t) + \mathcal{A}_{jk}(t)\right] &= -2i\mathcal{A}_{kj}(t) = B^*_{kj}(t), \\ -i\left[\mathcal{C}_{kj}(t) + \mathcal{B}_{jk}(t)\right] &= -2i\mathcal{C}_{kj}(t) = A_{kj}(t), \\ i\left[\mathcal{D}_{kj}(t) + \mathcal{D}_{jk}(t)\right] &= 2i\mathcal{D}_{kj}(t) = B_{kj}(t), \\ i\left[\mathcal{B}_{kj}(t) + \mathcal{C}_{jk}(t)\right] &= 2i\mathcal{B}_{kj}(t) = A^*_{kj}(t). \end{aligned}$$

Inserting the last coefficients into Eq. (18.27) and moving to the Schrodinger picture of quantum mechanics, one obtains the following expression for the effective Hamiltonian

$$\hat{H}_{\text{eff}}(t) = \frac{i}{2}\sum_{kj}\left[B^*_{jk}(t)\hat{a}_j^{in}\hat{a}_k^{in} - B_{jk}(t)\hat{a}_k^{in\dagger}\hat{a}_j^{in\dagger} + A^*_{jk}(t)\hat{a}_k^{in\dagger}\hat{a}_j^{in} - A_{jk}(t)\hat{a}_j^{in\dagger}\hat{a}_k^{in}\right], \tag{18.31}$$

where $\hat{a}_k^{in} \equiv \hat{a}_k(t=0)$ and $\hat{a}_k^{in\dagger} \equiv \hat{a}_k^\dagger(t=0)$ and the Heisenberg annihilation (and creations) operator is defined as $\hat{a}_k(t) = \hat{U}_S^\dagger(t)\hat{a}_k^{in}\hat{U}_S(t)$, with $\hat{U}_S(t)$ being the time evolution operator generated by the last Hamiltonian.

Here, we can clearly see the existence of two different contributions. The terms containing the coefficients B^*_{kj} and B_{kj} govern the process of creation and annihilation of pairs of particles, while the ones proportional to A^*_{kj} and A_{kj} are responsible for scattering of particles between distinct modes.

18.3.3 Long-time Bogoliubov coefficients

In Eq. (18.25), the first order Bogoliubov coefficients were derived from the dynamics of the instantaneous operators $\hat{a}_k(t)$ and $\hat{a}^\dagger_k(t)$. Although straightforward, such naive perturbation theory is in general plagued with secularities when the system exhibit some kind of resonance behaviour, as in the important case of oscillatory motions. For this reason, perturbative expressions obtained from the effective Hamiltonian (18.31) are generally constrained to the short-time regime. In this subsection we shall derive expressions for the Bogoliubov coefficients, in the special case when the cavity performs parametric oscillating, that are valid in all times.

For $t > 0$, when the mirror starts to move, the quantum field can still be decomposed in terms of the initial operators $\hat{a}^{in}_k$ and $\hat{a}^{in\dagger}_k$ in the form

$$\hat{\phi}(x, t > 0) = \sum_k \frac{1}{\sqrt{2\omega^{in}_k}} \left[\hat{a}^{in}_k f_k(x,t) + \hat{a}^{in\dagger}_k f^*_k(x,t)\right], \tag{18.32}$$

as long as the new set of mode functions $\{f_k(x,t)\}$ satisfies the conditions: (i) the wave equation (18.18), (ii) the time-dependent boundary condition (18.18), and (iii) the initial condition $f_k(x,0) = \sin(\omega^{it}_k x)\exp -i\omega^{it}_k t$. In this regard, we proceed by expanding the mode function in a series with respect to an *instantaneous basis* as

$$f_k(x,t) = \sum_j Q^{(k)}_j(t)\varphi_j(x,t), \tag{18.33}$$

where the Fourier coefficients $Q^{(k)}_j(t)$ must satisfy the differential equation

$$\ddot{Q}^{(k)}_j + \omega^2_j(t)Q^{(k)}_j = 2\lambda(t)\sum_{m\neq k} g_{km}\dot{Q}^{(k)}_m + \dot{\lambda}(t)\sum_{m\neq k} g_{km}Q^{(k)}_m + \mathcal{O}(\lambda^2) \tag{18.34}$$

$$\text{with} \quad g_{jk} = (-1)^{j-k}\frac{2kj}{j^2-k^2}, \tag{18.35}$$

and $\lambda(t) = \dot{L}(t)/L(t)$, together with the initial conditions

$$Q^{(k)}_j(0) = \delta_{jk}, \qquad \dot{Q}^{(k)}_j(0) = -i\omega^{\rm in}_k\delta_{kj}. \tag{18.36}$$

To make the problem more amenable we can turn to the special case of parametric resonance where the second mirror oscillates with small amplitude at twice the fundamental unperturbed field frequency. Such consideration can be achieved by imposing the following prescribe trajectory

$$L(t) = L_0\left[1 + \epsilon\sin\left(2\omega^{\rm in}_1 t\right)\right]. \tag{18.37}$$

By considering the mirror to return to its initial position L_0 after a finitee period of time T ($\omega^{\rm in}_k = \omega^{\rm out}_k = \omega_k$), the right-hand side of Eq. (18.34) vanishes and we are allow to write

$$Q_j^{(k)}(t \geq T) = \sqrt{\frac{\omega_k}{\omega_j}} \left(\alpha_{kj} e^{-i\omega_j t} + \beta_{kj} e^{i\omega_j t}\right), \tag{18.38}$$

where α_{kj} and β_{kj} stands for the Bogoliubov coefficients defined at time $t \geq T$.

Due to the weakly perturbed nature of the mirror oscillations (18.37), we search for solutions for the Fourier coefficient $Q_j^{(k)}(t)$ by allowing the Bogoliubov coefficients in Eq. (18.38) to be represented by slowly varying functions on time, i.e., with $\dot{\alpha}_{kj}, \dot{\beta}_{kj} \sim \epsilon$. By inserting Eq. (18.38) into Eq. (18.34), discarding terms which are proportional to ϵ^2 (as $\ddot{\alpha}_{kj}, \ddot{\beta}_{kj}$ and λ^2) and using the method of slowly varying amplitudes [63], we can obtain a set of coupled first order differential equations with time independent coefficients in terms of α_{kj} and β_{kj}. For the case of $k = 1$, such set of equations takes the following form [64]

$$\frac{\mathrm{d}\alpha_{1\mathrm{j}}}{\mathrm{d}\tau} = -\sqrt{3}\alpha_{3j} - \beta_{1j} \quad \text{and} \quad \frac{\mathrm{d}\beta_{1\mathrm{j}}}{\mathrm{d}\tau} = -\alpha_{1j} - \sqrt{3}\beta_{3j}, \tag{18.39}$$

while in the case of $k > 2$ one have

$$\frac{\mathrm{d}\alpha_{\mathrm{kj}}}{\mathrm{d}\tau} = \sqrt{k(k-2)}\alpha_{(k-2),j} - \sqrt{k(k+2)}\alpha_{(k+2),j}, \tag{18.40a}$$

$$\frac{\mathrm{d}\beta_{\mathrm{kj}}}{\mathrm{d}\tau} = \sqrt{k(k-2)}\beta_{(k-2),j} - \sqrt{k(k+2)}\beta_{(k+2),j}. \tag{18.40b}$$

Furthermore, as a consequence of the initial conditions $\alpha_{kj}(0) = \delta_{kj}$ and $\beta_{kj}(0) = 0$, one can show that all the coefficients with at least one even index must vanish.

The set of equations (18.39) and (18.40) were solved exactly in Ref. [65] and its asymptotic behaviour with index equal to 1 takes the following form [1]: for $\tau \ll 1$, their expressions read

$$\alpha_{1(2\mu+1)} = (\mu+1)K_\mu J_\mu \; \tau^\mu + \mathcal{O}(\tau^{\mu+2}), \tag{18.41a}$$

$$\beta_{1(2\mu+1)} = -K_\mu J_\mu \; \tau^{\mu+1} + \mathcal{O}(\tau^{\mu+3}), \tag{18.41b}$$

with $J_\mu = (2\mu)!/2^\mu(\mu!)^2$ and $K_\mu = (-1)^\mu\sqrt{2\mu+1}/(\mu+1)$, whereas for $\tau \gg 1$

$$\alpha_{1(2\mu+1)} \approx \frac{2}{\pi}\frac{(-1)^\mu}{\sqrt{2\mu+1}}, \quad \text{and} \quad \beta_{1(2\mu+1)} \approx -\frac{2}{\pi}\frac{(-1)^\mu}{\sqrt{2\mu+1}}, \quad \text{with} \quad (\mu = 0, 1, 2, \dots). \tag{18.42}$$

18.4 Entropy Production in the DCE

With all the ingredients at hands we can begin studying the irreversibility associated with the DCE in terms of the production of thermodynamic entropy. As discussed earlier, for this endeavour, we shall consider the diagonal entropy [39]

$$S_d(\hat{\rho}) = -\sum_{\boldsymbol{n}} \rho_{\boldsymbol{nn}} \ln \rho_{\boldsymbol{nn}}, \tag{18.43}$$

as the main figure of merit, where $\rho_{\boldsymbol{nn}} = \langle \mathrm{in}; \boldsymbol{n}| \, \hat{\rho} \, |\boldsymbol{n}; \mathrm{in}\rangle$ are the diagonal elements of the system's density operator in the initial energy eigenbasis.

[1] Later, when calculating the expression for the thermodynamic entropy for a particular mode of the field, we will be able to show that in the case of parametric oscillation, the reduced diagonal entropy depends only on the Bogoliubov coefficients with one of its index equals to 1.

18.4.1 Effective Hamiltonian approach

In order to analyse the thermodynamic entropy production within the proposed scheme, one first needs to obtain an explicit expression for the system's density operator $\hat{\rho}$ after the cavity returns to its stationary configuration. A first intuitive approach is to consider the effective Hamiltonian (18.31) derived in the previous chapter to evolve $\hat{\rho}$ through the Liouville–von Neumann equation

$$\dot{\hat{\rho}}(t) = -i\left[\hat{H}_{\text{eff}}(t), \hat{\rho}(t)\right]. \tag{18.44}$$

Conversely, the complex structure of the effective Hamiltonian poses inherent challenges in solving Eq. (18.44). To circumvent this problem, we opt to narrow our focus to the subset of cavity configurations characterized by weakly perturbed trajectories, such as in

$$L(t) = L_0\left[1 + \epsilon\xi(t)\right], \tag{18.45a}$$

with $\xi(t)$ being an arbitrary function of order unity —as well as its first time derivative— while $\epsilon \ll 1$ is a small amplitude. By imposing the mirror trajectories (18.45), we expect the corrections to the solvable part of the field (free part of the Hamiltonian) to be sufficiently weak, allowing for the application of perturbation theory.

To advance in the perturbation approach, a key insight is to acknowledge that the time-dependent coefficients defined in Eq. (18.22) are proportional to $\lambda(t) = \dot{L}(t)/L(t)$ and thus depend on the small amplitude ϵ through Eq. (18.45). Consequently, a formal solution to Eq. (18.44) up to second order in ϵ can be obtained by

$$\hat{\rho}(T) = \hat{\rho}(0) - i\int_0^T dt'\left[\hat{H}^I_{\text{int}}(t'), \hat{\rho}(0)\right] - \int_0^T dt' \int_0^{t'} dt''\left[\hat{H}^I_{\text{int}}(t'), \left[\hat{H}^I_{\text{int}}(t''), \hat{\rho}(0)\right]\right]. \tag{18.46}$$

Since we want to study the thermodynamics of the particle creation process due to the DCE, we are gonna be interested in the particular case in which the system is initially prepared in the initial vacuum state $\hat{\rho}(0) \equiv \hat{\rho}_0 = |0;\text{in}\rangle\langle\text{in};0|$. Another useful property for the expansion calculation, comes from the fact that the time-dependence in the Hamiltonian (18.31) is concentrated only on the coefficients. Consequently, one can write

$$\int_0^T dt'' H(t'') = \frac{i}{2}\sum_{nl}\left[\beta^{(1)*}_{kj}\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j - \beta^{(1)}_{ln}\hat{a}^{in}_k\hat{a}^{in}_j + \tilde{\alpha}^{(1)*}_{ln}\hat{a}^{in\dagger}_k\hat{a}^{in}_j - \tilde{\alpha}^{(1)}_{ln}\hat{a}^{in\dagger}_j\hat{a}^{in}_k\right]. \tag{18.47}$$

Inserting the last identity into the operator expansion (18.46) and after lengthy algebraic manipulations of commutators, one can obtain the following expression for the system's density operator (up to second order in ϵ)

$$\begin{aligned}
\hat{\rho} \;=\; & \hat{\rho}_0 - \tfrac{1}{2}\sum_{kj}\Bigg\{\beta^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right) - \tfrac{1}{4}\sum_{nm}\Big[\beta^{(1)}_{mn}\beta^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\hat{a}^{in}_m\hat{a}^{in}_n\right) \\
& -\beta^{(1)}_{mn}\beta^{(1)*}_{kj}\left(\hat{a}^{in}_m\hat{a}^{in}_n\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right) + \beta^{(1)*}_{mn}\beta^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_m\hat{a}^{in\dagger}_n\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right) \\
& +2\tilde{\alpha}^{(1)*}_{mn}\beta^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_m\hat{a}^{in}_n\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right)\Big] + \text{H.c.}\Bigg\},
\end{aligned} \tag{18.48}$$

where "H.c." stands for Hermitian conjugate.

With the last expression we can directly calculate the number of particles created inside the cavity due do the DCE, taking the following form

$$\begin{aligned} N(T) &= \mathrm{Tr}\left\{\sum_k \hat{\rho}(T)\hat{a}_k^{in\dagger}\hat{a}_k^{in}\right\} = \frac{1}{4}\Re\sum_{kk'jmn}\beta_{mn}^{(1)}\beta_{k'j}^{(1)*}\,\mathrm{Tr}\left\{\hat{a}_{k'}^{in\dagger}\hat{a}_j^{in\dagger}\hat{\rho}_0\hat{a}_m^{in}\hat{a}_n^{in}\hat{a}_k^{in\dagger}\hat{a}_k^{in}\right\} \\ &= \sum_{kj}|\beta_{kj}^{(1)}|^2, \end{aligned} \tag{18.49}$$

in agreement with Eq. (18.26), thus showing the consistency of our calculations. Moreover, one can obtain explicit expressions for the diagonal elements of the density operator in the initial energy basis directly from Eq. (18.48). Those elements come from three contributions

$$\langle 0|\,\hat{\rho}(T)\,|0\rangle = \langle 0|\,\hat{\rho}(0)\,|0\rangle - \frac{1}{4}\Re\sum_{kjmn}\beta_{mn}^{(1)}\beta_{kj}^{(1)*}\,\langle 0|\,\hat{a}_m^{in}\hat{a}_n^{in}\hat{a}_k^{in\dagger}\hat{a}_j^{in\dagger}\hat{\rho}(0)\,|0\rangle = 1 - \frac{1}{2}N(T), \tag{18.50a}$$

$$\langle 2_k|\,\hat{\rho}(T)\,|2_k\rangle = \frac{1}{4}\Re\sum_{k'jmn}\beta_{mn}^{(1)}\beta_{k'j}^{(1)*}\,\langle 2_k|\,\hat{a}_{k'}^{in\dagger}\hat{a}_j^{in\dagger}\hat{\rho}(0)\hat{a}_m^{in}\hat{a}_n^{in}\,|2_k\rangle = \frac{1}{2}|\beta_{kk}^{(1)}|^2, \tag{18.50b}$$

$$\langle 1_k, 1_j|\,\hat{\rho}(T)\,|1_k, 1_j\rangle = \frac{1}{4}\Re\sum_{k'j'mn}\beta_{mn}^{(1)}\beta_{k'j'}^{(1)*}\,\langle 1_k, 1_j|\,\hat{a}_{k'}^{in\dagger}\hat{a}_{j'}^{in\dagger}\hat{\rho}(0)\hat{a}_m^{in}\hat{a}_n^{in}\,|1_k, 1_j\rangle = |\beta_{kj}^{(1)}|^2. \tag{18.50c}$$

With the help of the last expressions we are now ready to discuss the entropy production due to the particle creation process.

Entropy production

In terms of the diagonal contributions of $\hat{\rho}$ given by Eqs. (18.50), the diagonal entropy S_d for the system can be obtained as[2]

$$\begin{aligned} S_d(T) = &-\langle 0|\,\hat{\rho}\,|0\rangle \ln\langle 0|\,\hat{\rho}\,|0\rangle - \sum_k \langle 2_k|\,\hat{\rho}\,|2_k\rangle \ln\langle 2_k|\,\hat{\rho}\,|2_k\rangle \\ &- \tfrac{1}{2}\sum_{kj}\langle 1_k, 1_j|\,\hat{\rho}\,|1_k, 1_j\rangle \ln\langle 1_k, 1_j|\,\hat{\rho}\,|1_k, 1_j\rangle \\ = &-\left[1 - \tfrac{1}{2}N(T)\right]\ln\left[1 - \tfrac{1}{2}N(T)\right] - \sum_{kj}\tfrac{1}{2}|\beta_{kj}(T)|^2\ln(1 - \tfrac{1}{2}\delta_{kj})|\beta_{kj}(T)|^2, \end{aligned} \tag{18.51}$$

which is the expression for the system's thermodynamics entropy at the second order of the perturbation parameter ϵ (associated to weakly perturbed mirror trajectories).

An immediate aspect encountered in Eq. (18.51) is related to its scaling behaviour with the number of particles created in the cavity as its moves. This is also explicit in its dependence on the $|\beta_{kj}|^2$, which is interpreted as the contribution of the number of particles in the k-th mode due to the j-th mode of the initial field. Such link between

[2] The 1/2 factor in the last term is employed to prevent the over-counting of the diagonal contributions $\langle 1_k, 1_j|\,\hat{\rho}\,|1_k, 1_j\rangle$ when summing over k and j, i.e., $\langle 1_k, 1_j|\,\hat{\rho}\,|1_k, 1_j\rangle$ is the same as $\langle 1_j, 1_k|\,\hat{\rho}\,|1_j, 1_k\rangle$.

irreversibility and the number of particle created in the process is consistent with the well established result that the DCE can only occurs if the field in perturbed in a non-adiabatic manner, *i.e.*, if the mirror positions changes rapidly enough to prevent the field to readjust instantaneously.

Another insight given by Eq. (18.51) to the interpretation of the thermodynamic entropy production is related in how the initial vacuum state with a definite energy value is transformed in a coherent superposition of excited states as time passes. In order to see this, we consider the relative entropy of coherence [66]

$$C(\hat{\rho}) = S_{vN}(\hat{\rho}_d) - S_{vN}(\hat{\rho}), \tag{18.52}$$

where $\hat{\rho}_d = \sum_i \rho_{ii} |i\rangle \langle i|$ are the diagonal contributions for the density operator in the basis $\{|i\rangle\}$. Expression (18.52) quantifies the amount of quantum coherence generated in the chosen basis. By picking up the initial energy eigenbasis to measure the amount of coherence generated throughout the cavity motion, we directly obtain that $S_{vN}(\hat{\rho}_d) = S_d(\hat{\rho})$. Since the system's evolution is unitary and the initial state is pure, we have $S_{vN}(\hat{\rho}) = 0$, thus implying

$$C(\hat{\rho}) = S_d(T). \tag{18.53}$$

It is important to remark that, differently from Eq. (18.51), expression (18.53) is valid in general, being independent of the perturbation theory previously used.

A consequence of the last result is that we expect to observe irreversibility (positive entropy production) for every process that creates quantum coherence in the system's energy eigenbasis. Thus, reversibility must be defined in terms of those process performed sufficiently slow to not induce transitions among the energy eigenstates. Such conclusion is in perfect agreement with the discussions presented in Refs. [39–41, 43], where both entropy production and heat are traced to processes whose net result is generation of coherence.

Oscillating mirror

To illustrate our results we consider the specific case in which the second moving mirror performs harmonic oscillations with the form

$$\xi(t) = \sin(\omega_p t), \tag{18.54}$$

where $\omega_p = p\omega_1^{in}$ is an integral multiple of the first unperturbed field frequency. Introducing the last oscillating function Eq. (18.54) into Eq. (18.25) and using $\omega_{kj} = \omega_k^{in} + \omega_j^{in}$, we obtain

$$\begin{aligned}
\beta_{jk} &= (-1)^{j-k} \epsilon \omega_p \frac{\sqrt{jk}}{k+j} e^{i\omega_k^{in} T} \int_0^T \mathrm{dt'} \cos(\omega_\mathrm{p} \mathrm{t'}) \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{kj}}\mathrm{t'}} \\
&= (-1)^{j-k} \epsilon \omega_p \frac{\sqrt{jk}}{k+j} e^{i\omega_k^{in} T} \begin{cases} \frac{2\omega_{kj}}{\omega_{kj}^2 - \omega_p^2} \sin\left(\frac{\omega_{kj} T}{2}\right) e^{\frac{i}{2}\omega_{kj} T} & \text{for} \quad \omega_p \neq \omega_{kj}, \\ \frac{1}{2} T & \text{for} \quad \omega_p = \omega_{kj}, \end{cases}
\end{aligned}$$

where we have assumed the circumstance in which the second mirror returns to its initial position at time $t = T$ after performing a certain number of complete cycles ($\omega_p T = 2\pi m$ with $m = 1, 2, \ldots$). If we apply a rotating-wave approximation on the previous equation, where we ignore all the fast oscillatory terms associated with the mode $\omega_p \neq \omega_k + \omega_j$, one can obtain

$$\beta_{jk} = (-1)^{j-k} \frac{\sqrt{jk}}{k+j} \frac{\epsilon \omega_p T}{2} e^{i\omega_k^{in} T} \delta_{j,(p-k)}, \quad \text{for} \quad k = 1, \ldots, p-1. \tag{18.55}$$

Using the last Eq. (18.55) one can compute the number of particles created inside the moving cavity as simply

$$N(\tau) = \sum_{jk} \left|\beta_{jk}(\tau)\right|^2 = p^2\tau^2 \sum_{kj} \frac{jk}{(k+j)^2} \delta_{j,(p-k)} = \tau^2 \sum_{k=1}^{p-1} (p-k)k = \frac{p(p^2-1)}{6}\tau^2, \tag{18.56}$$

where for conciseness we have introduced the dimensionless time $\tau = \frac{1}{2}\epsilon\omega_1^{in} T$.

The first noticeable aspect is that expression (18.56) is its full agreement with Ref. [65]. Another important remark to observe is that the resonance condition (18.54) introduce introduces secular terms proportional to orders of ϵT when applied the naive perturbation theory outlined in Eq. (18.46). Such inconvenience can be seen more directly if one consider the time scale is of order $T \sim \frac{1}{\epsilon} \gg 1$. In this case, perturbation theory breaks down and subsequent terms become greater than their predecessors. Consequently, expression (18.56) represents a reliable approximation for the number of particles only when $\tau \ll 1$.

Using Eq. (18.56) one directly obtain in the case of the oscillatory mirror configuration the followin expression for the thermodynamic entropy production

$$S_d(\tau) \quad = \quad \frac{1}{2}N(\tau)\left[1 - \ln\frac{1}{2}N(\tau) + \ln\frac{p(p^2-1)}{6} - \frac{6\,\mathrm{v}(p)}{p(p^2-1)}\right], \tag{18.57}$$

where

$$\mathrm{v}(p) = \sum_{k=1}^{p-1} (p-k)k\ln(p-k)k.$$

More specifically, figure 18.1 shows the behaviour of the diagonal entropy produced inside the cavity for the considered specific mirror motion. As it is clear from the figure, entropy will be produced in the field for every value of the mirror frequency p, except for $p = 1$, where the number of created particles vanishes.

In resume, the approach based on the effective Hamiltonian used here allowed us to calculate the system's entropy production by evolving the density operator in terms of its complete mode structure. Such method lead us to establishment of a profound connection between the thermodynamic entropy production and the generation of quantum coherence in the field energy basis. For the next section we rely on the evolution of the reduced density operator in terms of the distinct properties exhibited by Gaussian states.

18.4.2 Gaussian state approach

As we could see, the analysis for the entropy production introduced in the last section was constrained to the short-time regime. Here we present a distinct technique based on the evolution of Gaussian state that will allow us to analyse the diagonal entropy expression for a given mode in all times. Furthermore, such method will enable us to relate the field dynamics with that of entanglement between the considered mode and all the others.

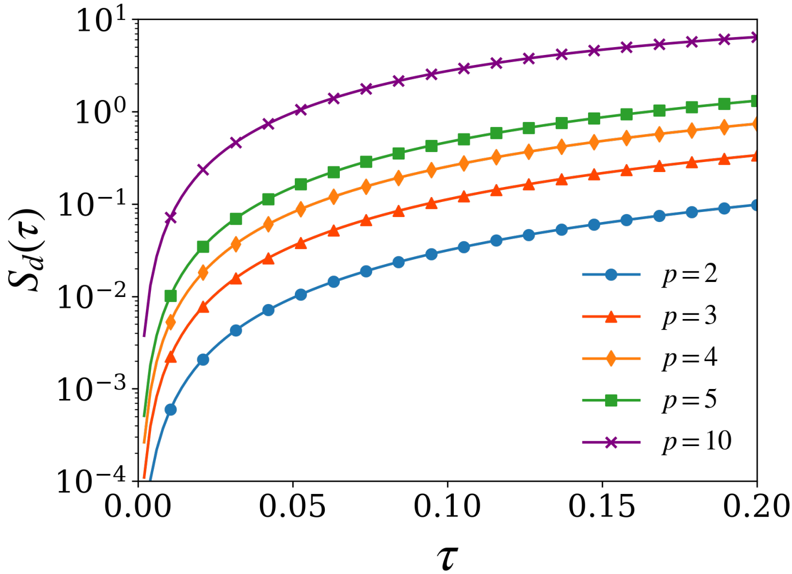

Figure 18.1: **Entropy production.**The diagonal entropy expression is given by Eq. (18.57) is plotted in terms of the dimensionless time τ for distinct values of the mirror frequency. It is important to observed that each point of the graphic represents the thermodynamical entropy for a given instant of time τ as if the cavity have stopped with the stantaneous cavity length $L(\tau)$.

Basic elements in the theory of Gaussian states

Let us start by considering the field quadrature operators $\hat{q}_k$ and $\hat{p}_k$ in the time interval *out*: $t \geq T$, *i.e.*, when the cavity has already returned to its static configuration. In the Schrödinger picture, their expressions reads

$$\hat{q}_k = \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_k + \hat{a}_k^\dagger\right), \quad \text{and} \quad \hat{p}_k = -i\sqrt{\frac{\omega_k}{2}} \left(\hat{a}_k^\dagger - \hat{a}_k\right), \tag{18.58}$$

where $\hat{a}_k \equiv \hat{a}_k(t = T)$, $\hat{a}_k^\dagger \equiv \hat{a}_k^\dagger(t = T)$, $\omega_k = \omega_k(T)$ and both operators satisfies the following commutation relations

$$\left[\hat{q}_k, \hat{p}_j\right] = i\delta_{kj} \quad \text{and} \quad \left[\hat{q}_k, \hat{q}_j\right] = \left[\hat{p}_k, \hat{p}_j\right] = 0. \tag{18.59}$$

Although we are dealing with an infinite dimensional system, for presentation purpose, we will consider the description of only a finite number N of modes. To this end, a more compact mathematical formulation for the system can be obtained with the introduction of the vector

$$\hat{\mathbf{R}} = \bigoplus_{k=1}^{N} \hat{\mathbf{r}}_k \qquad \text{with} \quad \hat{\mathbf{r}}_k = \frac{1}{\sqrt{\omega_k}} (\omega_k \hat{q}_k, \hat{p}_k)^T, \tag{18.60}$$

where the commutation relations are now summarised to a single expression

$$\left[\hat{R}_k, \hat{R}_j\right] = i\Omega_{kj}, \quad \text{with} \quad \mathbf{\Omega} = \bigoplus_{k=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{18.61}$$

with the orthogonal matrix $\mathbf{\Omega}$ defining the system's simplectic form.

In this context the description of the quantum state $\hat{\rho}$ can be obtained with the introduction of the Wigner quasi-probability function defined as

$$W(\mathbf{R}) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^{2N}} \mathrm{d}^{2\mathrm{N}}\mathbf{R} \mathrm{e}^{\mathrm{i}\mathbf{R}^{\mathrm{T}}\mathbf{\Omega}\hat{\mathbf{R}}} \mathrm{Tr}\left\{\hat{\rho}\hat{\mathrm{D}}(-\mathbf{R})\right\} \quad \text{for} \quad \hat{\mathrm{D}}(\mathbf{R}) = \bigotimes_{\mathrm{k}=1}^{\mathrm{N}} \mathrm{e}^{\mathbf{i}\mathbf{r}_{\mathrm{k}}^{\mathrm{T}}\mathbf{\Omega}\hat{\mathbf{r}_{\mathbf{k}}}}, \tag{18.62}$$

with $\mathbf{r}_k$ being the eigenvalues of $\hat{\mathbf{r}}_k$, as defined in Eq. (18.60) and

$$\mathrm{d}^{2\mathrm{N}}\mathbf{R} = \mathrm{dq}_1\mathrm{dp}_1 \ldots \mathrm{dq}_\mathrm{N}\mathrm{dp}_\mathrm{N}.$$

In particular, a N-mode state described by a density operator $\hat{\rho}$ is called a *Gaussian state* if its characteristic function is Gaussian, namely if

$$\chi_G(\mathbf{R}) = \exp -\frac{\mathbf{1}}{\mathbf{2}}\mathbf{R}^{\mathbf{T}}\mathbf{\Omega}\mathbf{\Sigma}\mathbf{\Omega}^{\mathbf{T}}\mathbf{R} - \mathbf{i}\mathbf{R}^{\mathbf{T}}\mathbf{\Omega}\langle\hat{\mathbf{R}}\rangle, \tag{18.63}$$

where $\mathbf{\Sigma}$ is the covariance matrix with elements

$$\Sigma_{kj} := \mathrm{Cov}(\hat{R}_k, \hat{R}_j) := \frac{1}{2}\langle\hat{R}_k\hat{R}_j + \hat{R}_j\hat{R}_k\rangle - \langle\hat{R}_k\rangle\langle\hat{R}_j\rangle, \tag{18.64}$$

from the standard definition of covariance $\mathrm{Cov}(\hat{A}, \hat{B}) = \frac{1}{2}\langle\triangle\hat{A}_m\triangle\hat{B}_m + \triangle\hat{B}_m\triangle\hat{A}_m\rangle$ for $\triangle\hat{A} = \hat{A} - \langle\hat{A}\rangle$. The correspondent Wigner function for a N-mode Gaussian state then takes the form

$$W_G(\mathbf{R}) = \frac{\mathbf{1}}{\sqrt{\det \mathbf{\Sigma}}} \exp -\frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{R} - \langle\hat{\mathbf{R}}\rangle\right)^{\mathbf{T}} \mathbf{\Sigma}^{-\mathbf{1}} \left(\mathbf{R} - \langle\hat{\mathbf{R}}\rangle\right), \tag{18.65}$$

which is therefore, fully characterized by the averages values of $\hat{\mathbf{R}}$ (called statistical first moment) and its covariance matrix $\mathbf{\Sigma}$ (the system's second statistical moment). Beyond this intrinsic simplicity of describing a system with the behavior with few parameters, the true importance of Gaussian states in our discussion comes from two theorems:

1. Unitary transformations that preserve the Gaussian character of a given state $\hat{\rho}_G$ (transforms any Gaussian state into another Gaussian state) are those generated by Hamiltonians which are at most quadratic in the canonical operator $\hat{\mathbf{R}}$, such as in [65, 67, 68]
$$\hat{H} = \frac{1}{2}\hat{\mathbf{R}}^T\mathbf{H}\hat{\mathbf{R}} + \hat{\mathbf{R}}^{\mathbf{T}}\hat{\mathbf{R}}, \tag{18.66}$$
where $\mathbf{H}$ is symmetric matrix.

2. If $\hat{H}$ is a quadratic Hamiltonian (as in Eq. (18.66)) with positive definite matrix $\mathbf{H}$, then any Gaussian state $\hat{\rho}_G$ can be written as a thermal state [67, 68]
$$\hat{\rho}_G = \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\left\{\mathrm{e}^{-\beta\hat{\mathrm{H}}}\right\}}, \tag{18.67}$$
where $\beta > 0$, which includes the limiting case $\beta \to \infty$ (ground state).

Reduced density operator

The last two results expressed by Eqs. (18.66) and (18.67) open to us the possibility of studying the entropy production in the DCE in terms of the evolution of Gaussian state. This occurs because in our description, the DCE is characterised (for all instants of time) in terms of a quadratic Hamiltonian on $\hat{a}_k$ and $\hat{a}_k^\dagger$. By initially preparing the system in the vacuum state $|0; in\rangle$ (which is a Gaussian state) one can show that the evolved state (induced by the particle creation process) must retain a Gaussian character, and therefore, be adequately specified by its covariant matrix and quadrature averages.

As exemplified in the short-time constraint under the Hamiltonian approach, the intrinsic complexity of the system discourage us from performing a multi-mode description through the referred Gaussian formalism. Consequently we choose to restrict attention to the time evolution of the density operator (its diagonal contributions in the energy eigenbasis) for a single-mode of the field.

More specifically, given a pure Gaussian state characterised by a density operator $\hat{\rho}(t)$ with the initial condition $\hat{\rho}(0) = |0; in\rangle\langle 0; in|$, the reduced density operator for a given mode m of the field can be computed as follows

$$\hat{\rho}^{(m)} = \mathrm{Tr}_{k\neq m}\,\hat{\rho}, \tag{18.68}$$

where we trace all the field modes expect $k = m$.

Under this considerations, the most general single-mode Gaussian state can be characterised in terms of the following Wigner function for the m-th field mode [69, 70]

$$W_G(\mathbf{r}_m) = \frac{1}{\sqrt{\det \boldsymbol{\Sigma}_m}} \exp -\frac{1}{2}\left(\mathbf{r}_m - \langle\hat{\mathbf{r}}_m\rangle\right)^T \boldsymbol{\Sigma}_m^{-1}\left(\mathbf{r}_m - \langle\hat{\mathbf{r}}_m\rangle\right), \tag{18.69}$$

where the m-th mode covariance matrix $\boldsymbol{\Sigma}_m$ stands for

$$\boldsymbol{\Sigma}_m \equiv \begin{pmatrix} \sigma_q^{(m)} & \sigma_{qp}^{(m)} \\ \sigma_{qp}^{(m)} & \sigma_p^{(m)} \end{pmatrix} \text{ with } \begin{cases} \sigma_q^{(m)} := \mathrm{Cov}(\hat{q}_m, \hat{q}_m) & = \omega_m\left[\langle\hat{q}_m^2\rangle - \langle\hat{q}_m\rangle^2\right], \\ \sigma_p^{(m)} := \mathrm{Cov}(\hat{q}_m, \hat{p}_m) & = \omega_m^{-1}\left[\langle\hat{p}_m^2\rangle - \langle\hat{p}_m\rangle^2\right] \\ \sigma_{qp}^{(m)} := \mathrm{Cov}(\hat{q}_m, \hat{p}_m) & = \frac{1}{2}\langle\hat{p}_m\hat{q}_m + \hat{q}_m\hat{p}_m\rangle - \langle\hat{q}_m\rangle\langle\hat{p}_m\rangle, \end{cases} \tag{18.70}$$

whose averages are computed in terms of initial state of the system.

The task to extract the diagonal contributions of the reduced density operator from our Wigner function (18.69) was already performed in Refs. [69, 70]. For the special case of an initially vacuum state $|0; in\rangle$, these diagonal terms can be written as functions of the covariance matrix elements [65]

$$\rho_m^{(n)} = \frac{2\left[\left(2\sigma_m^q - 1\right)\left(2\sigma_m^p - 1\right)\right]^{n/2}}{\left[\left(2\sigma_m^q + 1\right)\left(2\sigma_m^p + 1\right)\right]^{(n+1)/2}} \mathrm{P}_n\left(\frac{4\sigma_m^q\sigma_m^p - 1}{\sqrt{(4(\sigma_m^q)^2 - 1)(4(\sigma_m^p)^2 - 1)}}\right), \tag{18.71}$$

where P_n is the Legendre polynomial of order n and $\rho_m^{(n)} = \langle \mathrm{in}; n|\,\hat{\rho}_m\,|n; \mathrm{in}\rangle$ is the n-th diagonal element of the reduced density operator in the initial energy eigenbasis.

With expression (18.71) at hands, to calculate the diagonal entropy for a given field mode we only need to obtain explicit expressions for the field variances $\sigma_q^{(m)}$ and $\sigma_p^{(m)}$. To do this we begin considering the special case in which the cavity returns (after an interval of time T) to its initial cavity length L_0, such that the initial and final frequencies must be equal $\omega_k^{in} = \omega_k = \omega_k$. Expressing the quadrature operators (18.58) in terms of the initial operators $\hat{a}_k^{in}$ and $\hat{a}_k^{in\dagger}$ with the Bogoliubov transformations (18.23), the variances can be directly computed, resulting in

$$\sigma_q^{(m)} = \frac{1}{2}\sum_k |\alpha_{km} + \beta_{km}|^2, \quad \text{and} \quad \sigma_p^{(m)} = \frac{1}{2}\sum_k |\alpha_{km} - \beta_{km}|^2 \tag{18.72}$$

where m is an odd integer and the cross term $\sigma_{qp}^{(m)}$ is identically zero for the choice of the initial state. By taking the time derivatives of these last equations in respect to the dimensionless time τ one obtains

$$\left.\begin{matrix} \frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_q^{(m)} \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_p^{(m)} \end{matrix}\right\} = \sum_k \mathrm{Re}\left[(\alpha_{km} \pm \beta_{km})\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{km} \pm \frac{\mathrm{d}}{\mathrm{d}\tau}\beta_{km}\right)\right]. \tag{18.73}$$

Inserting the recursive relations(18.39) and (18.40) in Eqs. (18.73), and making the substitution $k \to k+2$ in one of its terms, its possible to verify that almost all the terms of the expression are cancelled, resulting in the following final form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_q^{(m)} = -|\alpha_{1m} + \beta_{1m}|^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_p^{(m)} = +|\alpha_{1m} - \beta_{1m}|^2, \tag{18.74}$$

which is dependent only through the Bogoliubov coefficients containing one of its index equal to 1 (as pointed out previously). Additionally, due to the definitions expressed in Eq. (18.58), one expects the differential equations (18.74) to respect the initial conditions $\sigma_q^{(m)}(0) = \sigma_p^{(m)}(0) = 1/2$. Let us now investigate their solutions in two different limits in time, the short and long-time regime.

Short-time regime

For the interval of time correspondent to the short-time regime, $\tau \ll 1$, the expressions for the system's quadrature variances can be obtained by introducing the Bogoliubov coefficients (18.41) into Eq. (18.74) and integrating the resulting differential equation, as in

$$\begin{aligned} \left.\begin{matrix} \sigma_q^{(2\mu+1)} \\ \sigma_p^{(2\mu+1)} \end{matrix}\right\} &= \frac{1}{2} \mp J_\mu^2 \int_0^\tau \mathrm{dt}\left[(2\mu+1)\mathrm{t}^{2\mu} \mp 2(\mu+1)\mathrm{K}_\mu^2 \mathrm{t}^{2\mu+1} + \mathcal{O}(\mathrm{t}^{2\mu+2})\right]\Big\} \\ &= \frac{1}{2} \mp \tau^{2\mu+1} J_\mu^2 \left[1 \mp K_\mu^2 \tau + \mathcal{O}(\tau^2)\right], \end{aligned} \tag{18.75}$$

where J_μ and K_μ are defined in Eq. (18.41).

By inserting expression (18.75) into Eq. (18.71) we can obtain the following expression for the diagonal components of the density operator

$$\begin{aligned} \rho_{nn}^{(2\mu+1)} &= (-1)^n i^n J_\mu^n \tau^{n(2\mu+1)} \left(1 - K_\mu^4 \tau^2\right)^{n/2} \left[1 - (n+1) J_\mu^2 \tau^{2\mu+2}\left(K_\mu^2 - \frac{1}{2} J_\mu^2 \tau^{2\mu}\right)\right] \\ &\quad \times P_n\left[i\tau\left(K_\mu^2 - J_\mu^2 \tau^{2\mu}\right)\right] + \mathcal{O}(\tau^{2\mu+3}). \end{aligned} \tag{18.76}$$

With the help of the last expression, one can calculate the reduced diagonal entropy at short-times in the $(2\mu+1)$-th field mode. If we consider only the second order contributions in τ we can obtain, for $\mu - 0$, we have

$$S_d^1(\tau \ll 1) = \frac{1}{2}N_1(\tau)\left[1 - \ln\frac{1}{2}N_1(\tau)\right],$$

whereas for any other value of μ, we obtain

$$S_d^{2\mu+1}(\tau \ll 1) = N_{2\mu+1}(\tau)\left[1 - \ln N_{2\mu+1}(\tau)\right] + \mathcal{O}(\tau^{2\mu+3}).$$

In both reduced diagonal entropy expressions, the number of particles created in a certain field mode $m - 2\mu + 1$ can be expressed as

$$N_{2\mu+1}(\tau) = K_\mu^2 J_\mu^2 \tau^{2\mu+2} + \mathcal{O}(\tau^{2\mu+3}).$$

As a result, at short times, the entropy for each mode, grows directly proportional to number of particles created at that field mode. Such outcome is consistent with the results presented with the Hamiltonian approach in the previous subsection.

Long-time regime

In respect to the long-time limit, $\tau \gg 1$, by introducing Eqs. (18.42) into Eqs. (18.74), one can find the time derivatives for the quadrature variances of the system as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_q^{(2\mu+1)} \approx 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_p^{(2\mu+1)} \approx \frac{16}{\pi^2(2\mu+1)}. \tag{18.77}$$

Solutions for the last expression takes the following form:

$$\sigma_q^{(2\mu+1)} \approx A_{2\mu+1} \quad \text{and} \quad \sigma_p^{(2\mu+1)} \approx \frac{16\tau}{\pi^2(2\mu+1)} + B_{2\mu+1}, \tag{18.78}$$

where $A_{2\mu+1}$ and $B_{2\mu+1}$ are constants of integration whose specific values varies from mode to mode and are dependent on the complete form of the Bogoliubov coefficients. However, one can establish certain behaviour for the last constant of integration: both quadrature variances start with the same value $1/2$ at $t = 0$ and evolve to assume distinct asymptotic behaviour at $\tau \gg 1$. More specifically, $\sigma_q^{(m)}$ decrease to a constant value whereas $\sigma_p^{(m)}$ increases almost linearly in time.

More specifically, explicit expression for the first two odd modes were derived in Ref. [65, 71] and are written in the form

$$\sigma_q^{(1)} \approx \frac{2}{\pi^2}; \qquad \sigma_p^{(1)} \approx \frac{16}{\pi^2}\tau + \frac{16}{\pi^2}\ln 2 - \frac{2}{\pi^2} \tag{18.79a}$$

$$\sigma_q^{(3)} \approx \frac{38}{9\pi^2}; \qquad \sigma_p^{(3)} \approx \frac{16}{3\pi^2}\tau + \frac{16}{3\pi^2}\ln 2 + \frac{10}{9\pi^2}. \tag{18.79b}$$

Therefore, by considering the general behaviour $\sigma_q^{(m)} \sim 1$ and $\sigma_p^{(m)} \sim \tau$, we compute the reduced density operator (single-mode expression) for the long-time regime by expanding Eq. (18.71) as

$$\rho_{nn}^{(m)}(\tau \gg 1) = C_n^{(m)} \; [\det \mathbf{\Sigma}_m(\tau)]^{-1/2} + \mathcal{O}(1/\tau) \tag{18.80}$$

where

$$C_n^{(m)} = \frac{1}{\sqrt{1+T_m}} \left(\frac{1-T_m}{\sqrt{1-T_m^2}} \right)^n \mathrm{P}_n \left(\frac{1}{\sqrt{1-T_m^2}} \right) \tag{18.81}$$

is a positive and real coefficient written in terms of $T_m = 1/2\sigma_m^q$.

Using Eq. (18.80), we can calculate the reduced diagonal entropy for the m-th field mode by decomposing the logarthim part in two terms

$$S_d^{(m)}(\tau \gg 1) \approx S_R^{(m)}(\tau) \sum_n \rho_{nn}^{(m)}(\tau) + [\det \boldsymbol{\Sigma}_m(\tau)]^{-1/2} \mathcal{S}(C_n^{(m)}), \tag{18.82}$$

where

$$S_R^{(m)}(\tau) = \frac{1}{2} \ln [\det \boldsymbol{\Sigma}_m(\tau)] \tag{18.83}$$

is the system's Rényi-2 entropy for the m-th mode and

$$\mathcal{S}(C_n^{(m)}) = -\sum_n C_n^{(m)} \ln C_n^{(m)}. \tag{18.84}$$

Before we interpret Eq. (18.82) in more details, let us check the convergence of each one of the summing parts. Considering the limiting case $n \gg 1$ for the Legendre polynomials

$$\mathrm{P}_n \left(\frac{1}{\sqrt{1-e^2}} \right) = \frac{1}{\sqrt{2\pi n e}} \frac{(1+e)^{\frac{n+1}{2}}}{(1-e)^{n/2}} + \mathcal{O}(1/n), \tag{18.85}$$

if one substitutes the last expression into Eq. (18.81), the coefficients $C_n^{(m)}$ can be show to satisfy the following asymptotic limit

$$C_{n \gg 1}^{(m)} = \frac{1}{\sqrt{\pi n}}. \tag{18.86}$$

Furthermore, by employing the integral test of convergence, one can shown that

$$\sum_n C_n^{(m)} < \sum_n C_{n\gg 1}^{(m)} < \frac{1}{\sqrt{\pi}} \int_0^{\mathcal{N}} \frac{dn}{\sqrt{n}} = \frac{2}{\sqrt{\pi}} \sqrt{\mathcal{N}},$$

where $\mathcal{N} \to \infty$ is the total number of modes. Thus, the sum over all the terms $\rho_{nn}^{(m)}$ (at the long-time regime) can be shown to satisfy

$$\sum_n \rho_{nn}^{(m)}(\tau \gg 1) = \sum_n C_n^{(m)} \left[\det \boldsymbol{\Sigma}_m\right]^{-1/2} < \frac{2}{\sqrt{\pi}} \frac{\sqrt{\mathcal{N}}}{\sqrt{\det \boldsymbol{\Sigma}_m}} \sim 1. \tag{18.87}$$

where we have considered that both $\mathcal{N}$ and $\det \boldsymbol{\Sigma}_m$ grow with the same order of magnitude in the long-time regime (since $\det \boldsymbol{\Sigma}_m \sim \tau$ for $\tau \to \infty$ since $\sigma_q^{(m)} \sim 1$ and $\sigma_p^{(m)} \sim \tau$). Indeed, we have estimated expression Eq. (18.87) only as prove of concept. By using the generating function for the Legendre polynomial, one can easily prove that $\mathrm{Tr}\left\{\hat{\rho}^{(\mathrm{m})}\right\}$ in terms of Eq. (18.71) is, as expected, exactly identical to unity.

As for the second term in Eq. (18.82), by inserting Eq. (18.86) into expression $S(C_n^{(m)})\det \boldsymbol{\Sigma}_m^{-1/2}$, one can prove that the correspondent summation can be restrict to respect

$$\begin{aligned}S(C_n^{(m)})\det \boldsymbol{\Sigma}_m^{-1/2} &< \frac{1}{2}\frac{1}{\sqrt{\det \boldsymbol{\Sigma}_m}}\int_0^{\mathcal{N}} \frac{\ln \pi n}{\sqrt{\pi n}}dn \\ &= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\frac{\sqrt{\mathcal{N}}}{\sqrt{\det \boldsymbol{\Sigma}_m}}\ln \pi\mathcal{N} - \frac{4}{\sqrt{\pi}}\frac{\sqrt{\mathcal{N}}}{\sqrt{\det \boldsymbol{\Sigma}_m}}\right) \sim \frac{1}{2}\ln(\pi\mathcal{N}) - 1,\end{aligned}$$

which means that the last series diverges logarithmically with the dimensionality $\mathcal{N}$ of the system. This basically occurs because we are dealing with a field theory and the number of the system's degree of freedom is infinity. Nevertheless, since we are considering the cavity motion to occur in a finite interval of time, for realistic situations with a limited amount of energy being injected we expect that only a finite number of modes to be excited and contribute to the last term. Since thermodynamic entropy is defined up to a multiplicative and additive constant, such term cannot be fundamental for the dynamics of the system's entropy.

As a result, the thermodynamic entropy for the m-th field mode can be identified in the long-time regime to be simply Rényi-2 entropy

$$S_d^{(m)}(\tau \gg 1) \approx S_R^{(m)}(\tau). \tag{18.88}$$

As the global state of the system is pure, $S_R^{(m)}(\tau)$ then needs to quantifies the amount of entanglement between the m-th mode and all the remaining ones. Eq. (18.82) the is telling us that the asymptotic behavior of the diagonal entropy is fundamentally determined by the generation of entanglement.

Using expressions given by Eq. (18.79) one can compute the system's thermodynamic entropy for the resonant mode $m = 1$,

$$S_d^{(1)}(\tau) \approx \frac{1}{2}\ln\left[\frac{4}{\pi^4}(8\tau + 2\ln 16 - 1)\right] \sim \frac{1}{2}\ln\left(\frac{32}{\pi^4}\tau\right),$$

which is in agreement with Ref. [36] [3] For the second odd mode, $m = 3$, we have

$$S_d^{(3)}(\tau) \approx \frac{1}{2}\ln\left[\frac{380}{81\pi^4}\left(\frac{24}{5}\tau + \frac{6}{5}\ln 16 + 1\right)\right] \sim \frac{1}{2}\ln\left(\frac{608}{27\pi^4}\tau\right).$$

18.5 Conclusions

From this study we have investigated the production of thermodynamic entropy in the DCE induced by the restriction of a quantum scalar field by a moving cavity. To calculate the relevant properties of our system we have used two distinct approaches: (i) from the dynamical evolution provided by an effective Hamiltonian we could connect the increase of entropy in the short-time limit with the generation of quantum coherence in the instantaneous energy eigenbasis; (ii) using the dynamics of the reduced density operator we

[3] Here, the argument in the Réniy-2 entropy differs from Ref. [36] by a factor of 4. This occurs because the variances defined in the last reference are twice as large as the ones in Eq. (18.70).

could also link the entropy growth in a given field mode (in all time regimes) with the generation of entanglement among the chosen mode and the rest of the field.

Combined, the two presented approaches gives us distinct but yet complementary insights in the dynamics of the thermodynamic entropy production in the DCE. Ultimately, the correspondent irreversible character in the phenomenon can the be determined by the generation of quantum coherence in the field mode basis and entanglement among such modes. From the properties of the diagonal entropy [39], due to the choice of a diagonal initial state we expect the entropy production, as well as the coherence and the entanglement generation, to only either increase or remain constant, but never decrease.

The previous results can be better understood by considering the process of *delocalization of energy* [39, 43], where an initial state with a definite energy value rapidly evolves to a coherent superposition of excited state due to the particle creation process (induced by the non-trivial boundary conditions). Although the system follow a unitary process, as we generally restricted to a limited set of possible measurements, one expects information about the energy state to become increasing inaccessible as it quickly spread throughout the system's Hilbert space. Such increase in the uncertainty on energy measurements linked to generation of quantum coherence is the main responsible to drive the irreversible dynamics of the system, as predicted by the non-vanishing value for the system's diagonal entropy production [39–41, 43]. The other side on the entropy production in the DCE is associated to the non-trivial coupling among the field's mode. As a result of the strong inter-mode interaction also predicts the generation of entanglement among such modes. With different parts of the system becoming strongly correlated from the entanglement production among the modes one also expects information about the energy values to become increasing difficult to recover. Reversibility can then be defined in the limit where the mirror motion is sufficiently slow so that no particle are created, are inter-mode scattered, or entropy is generated.

In retrospect, we think this work enhances the understanding of thermodynamic considerations in quantum field theories under non-trivial boundary conditions. The central message conveyed here is irreversibility in the DCE can be traced back to the strong inter-mode interaction which generate entanglement among different parts of the system and the (almost unavoidable) transitions between the instantaneous energy values (due to the generation of coherence) and .

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18.6 References

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