Chapter 16

Probability distributions describing quantum states

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16.1 Introduction

We formulate the possibility to describe quantum states by conventional probability distributions determining the density operators of these quantum states. Examples of quantum oscillator states and spin-1/2 states are considered, and the probability distributions determining the density matrices of the oscillator states and states of spin with given spin projections are explicitly presented.

Quantum-mechanical systems have the states, which are usually described either by complex wave functions [1] or by density matrices [2, 3]. There exist other possibilities to describe quantum states, using different quasidistributions like the Wigner function [4], Husimi function [5], and Glauber–Sudarshan function [6, 7]. In classical statistical mechanics, the system states are described by conventional probability distributions [8]. There were attempts to describe quantum states by conventional probability distributions, but they had difficulties due to the existence of uncertainty relation [9, 10]; the probability representation of quantum mechanics was constructed in [11, 12] after several decades of initial studies on the foundation of quantum mechanics [13].

Here, we formulate the approach, where the probability representation of quantum states is constructed on the examples of harmonic oscillator and spin-1/2 states.

In Section 2, we formulate the quantuzer–dequantizer operators for constructing the probability representation of quantum states. In Section 3, we consider the example of harmonic oscillator. Then, in Section 4, we study the example of qubit state (spin-1/2 state) and construct the probability distributions – alternatives of the density operators. We close our presentation by conclusions and provide the suggestions for future development of the probability representation of quantum mechanics.

16.2 Quantizer–dequantizer operators

In the conventional formulation of quantum mechanics, we use the notion of vectors $\mid \psi\rangle$ in the Hilbert space $\mathcal{H}$ and operators $\hat{\rho}$ called the density operators acting in this space. The wave function $\psi(x) = \langle x \mid \psi\rangle$ and the density matrix $\rho(x,x') = \langle x \mid \hat{\rho} \mid x'\rangle$ in the position representation are usual objects associated with states; they obey the quantum equations determining the system energy levels [13].

One can construct an invertible map of the operators $\hat{\rho}$ and other operators $\hat{A}$ acting in the Hilbert space $\mathcal{H}$ onto their symbols $f_\rho(\vec{X})$ and $f_A(\vec{X})$. These operators act on the functions $\psi(x)$, using the sets of operators $\hat{U}(\vec{X})$ and $\hat{D}(\vec{X})$ called dequantizers and quantizerz, respectively. Symbols of the operators satisfy the equations

$$f_A(\vec{X}) = \mathrm{Tr}\left(\hat{A}\hat{U}(\vec{X})\right), \tag{16.1}$$

$$\hat{A} = \int f_A(\vec{X})\hat{D}(\vec{X})\, d\vec{X}. \tag{16.2}$$

Since the map is invertible, information on the operator $\hat{A}$ is complete; also, information contained in symbol of the operator $\hat{A}$ is complete. If symbol of the density operator $\hat{\rho}$ is the probability distribution $f_\rho(\vec{X})$, the quantum state is described by symbol $f_\rho(\vec{X})$. All known representations of quantum states, such as the Wigner function [4], Husimi function [5], and Glauber–Sudarshan function [6, 7], are determined by specific pairs of quantizer–dequantizer operators. In the case, where the dequantizer operator has the properties of the density operator, its symbol $f_\rho(\vec{X})$ is the probability distribution [14, 15].

Thus, the problem of constructing the probability representation of quantum states is reduced to problem of finding the dequantizer operator $\hat{U}(\vec{X})$ and quantizer operator $\hat{D}(\vec{X})$, respectively, where $\hat{U}(\vec{X})$ has the properties of the density operator, namely, it is Hermitian, normalized $\mathrm{Tr}\,\hat{U}(\vec{X}) = 1$, and has nonnegative eigenvalues.

All representations of operators known in quantum mechanics, including density operators, are connected due to the relations associated with different kinds of quantizer–dequantizer operator pairs. We assume that, in addition to the quantizer–dequantizer operator pairs providing formulas (16.1) and (16.2), there exists the other pair $\hat{U}(\vec{Y})$ and $\hat{D}(\vec{Y})$ providing the relations

$$f_A(\vec{Y}) = \mathrm{Tr}\left(\hat{A}\hat{U}(\vec{Y})\right), \tag{16.3}$$

$$\hat{A} = \int f_A(\vec{Y})\hat{D}(\vec{Y})\, d\vec{Y}. \tag{16.4}$$

In view of (16.2)–(16.4), symbols of operator $\hat{A}$ are connected by integral transforms

$$f_A(\vec{Y}) = \int f_A(\vec{X}) \left[\mathrm{Tr}\,\hat{D}(\vec{X})\hat{U}(\vec{Y})\right] d\vec{X}. \tag{16.5}$$

Since all known representations, such as, for example, the Wigner function, are determined by the quantizer–dequantizer operator pairs, new probability representations should be connected with all known ones by integral transforms as well. As an example, we can present our work [11, 12], where probability distributions of oscillator states were connected with the Wigner function by Radon transform. After obtaining dequantizer operators with the properties of density matrices, one can construct the probability representation of quantum states. In the next section, we consider the first example of oscillator states.

16.3 Oscillator states in the probability representation

An important example of quantum-mechanical system is harmonic oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}; \tag{16.6}$$

here, we assume the oscillator mass $m = 1$ and the oscillator frequency $\omega = 1$. Its wave functions $\psi_n(x)$ satisfy the Schrödinger equation

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{x^2}{2}\right)\psi_n(x) = E_n\psi_n(x), \tag{16.7}$$

where the energy levels $E_n = n + 1/2$, Planck's constant $\hbar = 1$, and $\psi_n(x)$ is expressed in terms of Hermite polynomials as follows:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\,\pi^{-1/4}\,e^{-x/2}H_n(x). \tag{16.8}$$

Wave functions play important role in describing quantum phenomena.

Also, the wave function $\psi_\alpha(x)$ of coherent states $\langle x|\alpha\rangle$ [6, 7] satisfies the Schrödinger equation and the relation

$$\hat{a}\psi_\alpha(x) = \alpha\psi_\alpha(x); \qquad \hat{a} = 2^{-1/2}\left(\hat{q} + i\hat{p}\right) \tag{16.9}$$

and has the Gaussian form

$$\psi_\alpha(x) = \pi^{-1/4}\exp\left(-\frac{x^2}{2} + \sqrt{2}\alpha x - \frac{\alpha^2}{2} - \frac{|\alpha|^2}{2}\right). \tag{16.10}$$

To consider the probability representation of the oscillator state, we use the dequantizer operator $\hat{U}(\vec{X})$; $\vec{X} = (X, \mu, \nu)$ and the quantizer operator $\hat{D}(\vec{X})$; they read

$$\hat{U}(\vec{X}) = \delta\left(X\hat{\mathbf{1}} - \mu\hat{q} - \nu\hat{p}\right), \tag{16.11}$$

$$\hat{D}(\vec{X}) = \frac{1}{2\pi}\exp\left(iX\hat{\mathbf{1}} - i\mu\hat{q} - i\nu\hat{p}\right). \tag{16.12}$$

For complex wave function $\psi(x)$, symbol of the operator $\hat{\rho}_\psi(x, x') = \psi(x)\psi^*(x')$ is

$$w_{\rho_\psi}(X \mid \mu.\nu) = \mathrm{Tr}\left[\hat{\rho}_\psi\,\delta\left(X\hat{\mathbf{1}} - \mu\hat{p} - \nu\hat{q}\right)\right]; \tag{16.13}$$

it provides the possibility to reconstruct the density operator $\hat{\rho}$, in view of the relation

$$\hat{\rho} = \frac{1}{2\pi}\int w_\rho(X \mid \mu, \nu)\exp\left[i\left(X\hat{\mathbf{1}} - \mu\hat{q} - \nu\hat{p}\right)\right]\,dX\,d\mu\,d\nu. \tag{16.14}$$

For pure states $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$, symbol of the density operator called symplectic tomogram has the form [11],

$$w_\psi(X \mid \mu, \nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left[i\left(\frac{\mu}{2\nu}y^2 - \frac{X}{\nu}y\right)\right]\,dy\right|^2. \tag{16.15}$$

One can check that the oscillator tomograms are positive and normalized probability distributions; it means that

$$\int w_\psi(X \mid \mu,\nu)\,dX = 1. \tag{16.16}$$

Thus, for the ground state of the oscillator $\psi_0(x) = \pi^{-1/4}e^{-x^2/2}$, in view of (16.16), we arrive at the normal probability distribution

$$w_0(X \mid \mu,\nu) = \frac{\exp\left[-X^2/(\mu^2+\nu^2)\right]}{\sqrt{\pi(\mu^2+\nu^2)}}\,, \tag{16.17}$$

with zero mean value and dispersion equal to $(\mu^2+\nu^2)$. One can check that

$$\begin{aligned}\rho_0(x,x') &= \frac{1}{2\pi}\int \frac{\exp\left[-X^2/(\mu^2+\nu^2)\right]}{\sqrt{\pi(\mu^2+\nu^2)}} \\ &\quad\times\langle x \mid \exp\left[i\left(X\hat{\mathbf{1}}-\mu\hat{q}-\nu\hat{p}\right)\right] \mid x'\rangle\,dX\,d\mu\,d\nu \\ &= \pi^{-1/2}\exp\left[-\frac{x^2}{2}-\frac{x'^2}{2}\right].\end{aligned} \tag{16.18}$$

Tomogram (16.13) is the conditional probability distribution with real parameters μ and ν, which define the reference frames in the oscillator phase space with the property $X\hat{\mathbf{1}} = \mu\hat{q}+\nu\hat{p}$. An analogous relation in classical mechanics provides the possibility to introduce the tomographic representation [16, 17], using Radon transform [18] of the probability density $f(p,q)$ of classical particle

$$w_{\rm cl}(X \mid \mu,\nu) = \int f(p,q)\delta\left(X-\mu q-\nu p\right)\,dX\,d\mu\,d\nu. \tag{16.19}$$

The Radon transform of the probability distribution $f(q,p,t)$ describing the state of classical particle, including the particle trajectory $q(t) = F(q_0,p_0,t)$; $p(t) = \dot{q}(t)$, where q_0 and p_0 are the initial points in the phase space of the particle at time $t=0$, provides symplectic tomogram, which describes the behavior of the classical particle. In fact, one can show this behavior on an example of the free-particle motion with $q(t) = q_0 + p_0 t$; $p(t) = p_0$. The probability distribution $f(q,p,t) = \delta[q-(q_0+p_0t)]\delta(p-p_0)$ describes the motion of free particle with the initial position q_0 and the initial momentum p_0 at $t=0$. Then the classical-state tomogram of free-particle motion is given by Radon transform

$$\begin{aligned}w_{\rm cl}(X \mid \mu,\nu,t) &= \int f(p,q,t)\delta\left(X-\mu q-\nu p\right)dq\,dp \\ &= \delta\left[X-\mu(q_0+p_0t)-\nu p_0\right].\end{aligned} \tag{16.20}$$

It is important that inverse Radon transform reconstructs the probability distribution of the particle with given trajectory; on the example of free particle, this means that

$$\frac{1}{4\pi^2}\int w_{\rm cl}(X \mid \mu,\nu,t)\exp\left[i\left(X-\mu q-\nu p\right)\right]dX\,d\mu\,d\nu = f(p,q,t) \tag{16.21}$$

and, for free particle, we have

$$\begin{aligned}&\frac{1}{4\pi^2}\int \delta\left[X-\mu(q_0+p_0t)-\nu p_0\right] \\ &\times\exp\left[i\left(X-\mu q-\nu p\right)\right]dX\,d\mu\,d\nu \\ &= \delta\left[q-(q_0+p_0t)\right]\delta\left(p-p_0\right).\end{aligned} \tag{16.22}$$

Analogous relations of tomograms of classical particle and the written probability density can be checked for any trajectory satisfying the Newton equation [16, 17].

In the approach presented, the integrals of motion play an important role; analogous integrals of motion are available in the case of Hamiltonians, which are quadratic in positions and momenta [19, 20] – examples of such Hamiltonians are Hamiltonians of oscillator systems.

16.4 Qubit states in the probability representation

As an example of qubit states, let us consider the spin-1/2 states. To describe the observables $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$, namely, spin projections onto the x, y, and z axes, one can use the Pauli matrices, which read

$$\hat{S}_x = \frac{\hbar}{2}\,\sigma_x, \qquad \hat{S}_y = \frac{\hbar}{2}\,\sigma_y, \qquad \hat{S}_z = \frac{\hbar}{2}\,\sigma_z, \tag{16.23}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16.24}$$

The normalized eigenvectors $\mid \psi_x\rangle$, $\mid \psi_y\rangle$, and $\mid \psi_z\rangle$ of spin projections $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$, where we assume Plank's constant $\hbar = 1$, have the form

$$\mid \psi_x\rangle_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mid \psi_x\rangle_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{16.25}$$

$$\mid \psi_y\rangle_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mid \psi_y\rangle_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{16.26}$$

$$\mid \psi_z\rangle_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mid \psi_z\rangle_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{16.27}$$

They satisfy the following conditions:

$$\hat{S}_x \mid \psi_x\rangle_\pm = \pm\frac{1}{2} \mid \psi_x\rangle_\pm, \tag{16.28}$$

$$\hat{S}_y \mid \psi_y\rangle_\pm = \pm\frac{1}{2} \mid \psi_y\rangle_\pm, \tag{16.29}$$

$$\hat{S}_z \mid \psi_z\rangle_\pm = \pm\frac{1}{2} \mid \psi_z\rangle_\pm. \tag{16.30}$$

The density operators of the pure states (16.25)–(16.27) read

$$\hat{\rho}_{x+} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \qquad \hat{\rho}_{x-} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}, \tag{16.31}$$

$$\hat{\rho}_{y+} = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}, \qquad \hat{\rho}_{y-} = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}, \tag{16.32}$$

$$\hat{\rho}_{z+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\rho}_{z-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{16.33}$$

One can see that operators (16.31)–(16.33) have the following property:

$$\hat{\rho}_{x\pm}^{\dagger} = \hat{\rho}_{x\pm}, \quad \hat{\rho}_{y\pm}^{\dagger} = \hat{\rho}_{y\pm}, \quad \hat{\rho}_{z\pm}^{\dagger} = \hat{\rho}_{z\pm}. \tag{16.34}$$

The eigenvalues of these operators are equal one or zero, i.e., they are nonnegative; also, their trace is equal to one. If one has the density matrix of qubit state $\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$, which also has analogous properties: $\hat{\rho}^{\dagger} = \hat{\rho}$, $\mathrm{Tr}\,\hat{\rho} = 1$, and the eigenvalues of $\hat{\rho}$ are nonnegative, then, due to Born's rule [14, 15], we arrive at the numbers $\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{x\pm}\right)$, $\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{y\pm}\right)$, and $\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{z\pm}\right)$ in the form of conditional probabilities; they read

$$p_1 = w(+1/2 \mid 1) = \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{x+}\right), \quad 1 - p_1 = w(-1/2 \mid 1) = \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{x-}\right), \tag{16.35}$$
$$p_2 = w(+1/2 \mid 2) = \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{y+}\right), \quad 1 - p_2 = w(-1/2 \mid 2) = \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{y-}\right), \tag{16.36}$$
$$p_3 = w(+1/2 \mid 3) = \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{z+}\right), \quad 1 - p_3 = w(-1/2 \mid 3) = \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_{z-}\right). \tag{16.37}$$

We see that the conditional probability distributions $w(X \mid j)$, with $X = \pm 1/2$ and $j = 1, 2, 3$, satisfy the property

$$\sum_{X=-1/2}^{1/2} w(X \mid j) = 1.$$

Thus, the physical meaning of the qubit density matrix elements is associated with the conditional probability distribution describing the spin projections on the directions in the space.

The physical meaning of the probability distribution $w(X \mid j)$ is the probability to have, in the state with the density operator $\hat{\rho}$, the probability for spin projection $\pm 1/2$ onto the x, y, and z axes if $j = 1, 2, 3$. Thus, random variable X is the spin projection onto the x, y, and z directions, with spin parameters $j = 1, 2, 3$ showing the spin directions.

In view of relations (16.35)–(16.37), we can obtain the dequantizers $\hat{U}(X \mid j)$ and they, in turn, provide the possibility to arrive at the probability distributions $w(X \mid j)$ for qubit state with the density operator $\hat{\rho} = \hat{\rho}(\pm 1/2 \mid j)$; they are

$$\hat{U}(+1/2 \mid 1) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \tag{16.38}$$

$$\hat{U}(-1/2 \mid 1) = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}, \tag{16.39}$$

$$\hat{U}(+1/2 \mid 2) = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}, \tag{16.40}$$

$$\hat{U}(-1/2 \mid 2) = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}, \tag{16.41}$$

$$\hat{U}(+1/2 \mid 3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{16.42}$$

$$\hat{U}(-1/2 \mid 3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{16.43}$$

The matrix elements of the state density matrix $\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ can be expressed in terms of probabilities (16.35)–(16.37); then the density matrix reads [21]

$$\hat{\rho} = \begin{pmatrix} p_3 & p_1 - (1/2) - i(p_2 - 1/2) \\ p_1 - (1/2) + i(p_2 - 1/3) & 1 - p_3 \end{pmatrix}. \qquad (16.44)$$

Taking into account that $\sum_{X=-1/2}^{1/2} w(X \mid j) = 1$ and introducing the parameters describing the differences of probabilities $\frac{1}{2}\left[w(+1/2 \mid j) - w(-1/2 \mid j)\right]$, namely,

$$\Delta_+(1) = (2p_1 - 1)/2; \quad \Delta_+(2) = (2p_2 - 1)/2; \quad \Delta_+(3) = (2p_3 - 1)/2, \qquad (16.45)$$

we arrive at the density matrix of the form

$$\hat{\rho} = \begin{pmatrix} (1/2) + \Delta(3) & \Delta(1) - i\Delta(2) \\ \Delta(1) + i\Delta(2) & (1/2) - \Delta(3) \end{pmatrix}. \qquad (16.46)$$

For spin pure states, $\hat{\rho}^2 = \hat{\rho}$ and the condition parameter satisfies the relation $|\Delta(j)| \leq 1/2$. For mixed states, the purity parameter $\mathrm{Tr}\,\hat{\rho}^2 < 1$. One can introduce the probability distribution $P(X, j)$ of random variables $X = \pm 1/2$; $j = 1, 2, 3$ as $P(X, j) = w(X \mid j)W(j)$, where the probability distribution $W(j) \geq 0$, $\sum_{j=1}^{3} W(j) = 1$, and $\sum_{X=-1/2}^{1/2} P(X, j) = W(j)$.

Another spin-1/2 system probability representation was constructed in [22].

The von Neumann entropy for this distribution in the probability representation reads

$$\begin{aligned} S &= -\mathrm{Tr}\,\hat{\rho}\ln\hat{\rho} = -\mathrm{Tr}\left[\begin{pmatrix} (1/2) + \Delta(3) & \Delta(1) - i\Delta(2) \\ (1/2)\Delta(1) + i\Delta(2) & (1/2) - \Delta(3) \end{pmatrix}\right. \\ &\quad \left.\times \ln\begin{pmatrix} (1/2) + \Delta(3) & \Delta(1) - i\Delta(2) \\ (1/2)\Delta(1) + i\Delta(2) & (1/2) - \Delta(3) \end{pmatrix}\right]. \end{aligned} \qquad (16.47)$$

Also, we can consider the Shannon entropy

$$H = -\sum_{j=1}^{3} \sum_{X=-1/2}^{1/2} w(X \mid j) \ln w(X \mid j). \qquad (16.48)$$

In the case of two qubits, one can construct the quantizer and dequantizer operators and study the entangled probability distributions of the system consisted of two spin-1/2 subsystems.

16.5 Two spin-1/2 systems

To generalize our construction of the quantizer and dequantizer operators for the case of two qubits [23, 24], we introduce the operator of the following form:

$$\hat{U}\left(X_1, X_1 \mid j_1, j_2\right) = \hat{U}\left(X_1 \mid j_1\right) \otimes \hat{U}\left(X_2 \mid j_2\right), \qquad (16.49)$$

where operators $\hat{U}\left(X_1 \mid j_1\right)$ and $\hat{U}\left(X_2 \mid j_2\right)$ are given by (16.38)–(16.43). By construction, these operators have the properties of density operators and, in view of this fact, the density operator $\hat{\rho}$ of two qubits (ququart) $\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}$ provides the conditional probability distribution determined by 36 probabilities

$$w\left(X_1, X_1 \mid j_1, j_2\right) = \mathrm{Tr}\left[\hat{\rho}\,\hat{U}\left(X_1, X_1 \mid j_1, j_2\right)\right], \tag{16.50}$$

where $X_1 = \pm 1/2$, $X_2 = \pm 1/2$, $j_1 = 1, 2, 3$, and $j_2 = 1, 2, 3$.

We introduce notation

$$\begin{aligned} w\left(+1/2, +1/2 \mid 3, 3\right) = p_{z\uparrow, z\uparrow}, \quad w\left(+1/2, -1/2 \mid 3, 3\right) = p_{z\uparrow, z\downarrow}, \\ w\left(-1/2, +1/2 \mid 3, 3\right) = p_{z\downarrow, z\uparrow}, \quad w\left(-1/2, -1/2 \mid 3, 3\right) = p_{z\downarrow, z\downarrow}. \end{aligned} \tag{16.51}$$

In this notation, the relation of the probabilities with density matrix of two-spin states explicitly reads

$$p_{z\uparrow, z\uparrow} = \mathrm{Tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right].$$

This means that

$$p_{z\uparrow, z\uparrow} = \rho_{11}. \tag{16.52}$$

Analogously, we give explicitly the matrix elements of the density matrix expressed in terms of the probabilities $p_{z\uparrow,z\uparrow}$, $p_{z\uparrow,z\downarrow}$, $p_{z\downarrow,z\uparrow}$, $p_{z\downarrow,z\downarrow}$, $p_{x\uparrow,x\uparrow}$, $p_{x\uparrow,x\downarrow}$, $p_{x\downarrow,x\uparrow}$, $p_{x\downarrow,\downarrow}$, $p_{y\uparrow,y\uparrow}$, $p_{y\uparrow,y\downarrow}$, $p_{y\downarrow,y\uparrow}$, $p_{y\downarrow,y\downarrow}$, $p_{z\uparrow,x\uparrow}$, $p_{x\uparrow,z\uparrow}$, $p_{x\uparrow,z\downarrow}$, $p_{z\downarrow,x\uparrow}$, $p_{z\uparrow,x\downarrow}$, $p_{x\downarrow,z\uparrow}$, $p_{x\downarrow,z\downarrow}$, $p_{z\downarrow,x\downarrow}$, $p_{z\uparrow,y\uparrow}$, $p_{y\uparrow,z\uparrow}$, $p_{y\uparrow,z\downarrow}$, $p_{z\downarrow,y\uparrow}$, $p_{z\uparrow,y\downarrow}$, $p_{y\downarrow,z\uparrow}$, $p_{y\downarrow,z\downarrow}$, $p_{z\downarrow,y\downarrow}$, $p_{y\uparrow,x\uparrow}$, $p_{x\uparrow,y\uparrow}$, $p_{x\uparrow,y\downarrow}$, $p_{y\downarrow,x\uparrow}$, $p_{y\uparrow,x\downarrow}$,

$p_{x\downarrow,y\uparrow}$, $p_{x\downarrow,y\downarrow}$, and $p_{y\downarrow,x\downarrow}$ as follows:

$$
\begin{aligned}
&\rho_{11} = p_{z\uparrow,z\uparrow},\\
&\rho_{22} = p_{z\uparrow,z\downarrow},\\
&\rho_{33} = p_{z\downarrow,z\uparrow},\\
&\rho_{44} = p_{z\downarrow,z\downarrow},\\
&\rho_{12} = \rho_{21}^{*} = p_{z\uparrow,x\uparrow} - \frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\uparrow,z\downarrow}\right)\\
&+i\left[\frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\uparrow,z\downarrow}\right) - p_{z\uparrow,y\uparrow}\right],\\
&\rho_{13} = \rho_{31}^{*} = p_{x\uparrow,z\uparrow} - \frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\downarrow,z\uparrow}\right)\\
&+i\left[\frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\uparrow,z\downarrow}\right) - p_{y\uparrow,z\uparrow}\right],\\
&\rho_{14} = \rho_{41}^{*} = \frac{1}{2}\left(p_{x\uparrow,x\uparrow} + p_{x\downarrow,x\downarrow} - p_{y\uparrow,y\uparrow} - p_{y\downarrow,y\downarrow}\right)\\
&\qquad\qquad + \frac{i}{2}\left(p_{y\uparrow,x\downarrow} + p_{y\downarrow,x\uparrow} + p_{x\uparrow,y\downarrow} + p_{x\downarrow,y\uparrow} - 1\right),\\
&\rho_{23} = \rho_{32}^{*} = \frac{1}{2}\left(p_{x\uparrow,x\uparrow} + p_{x\downarrow,x\downarrow} - p_{y\uparrow,y\downarrow} - p_{y\downarrow,y\uparrow}\right)\\
&\qquad\qquad + \frac{i}{2}\left(p_{y\uparrow,x\downarrow} + p_{y\downarrow,x\uparrow} - p_{x\uparrow,y\downarrow} - p_{x\downarrow,y\uparrow}\right),\\
&\rho_{24} = \rho_{42}^{*} = \frac{1}{2}\left(p_{z\uparrow,z\downarrow} + p_{z\downarrow,z\downarrow}\right) - p_{x\downarrow,z\downarrow}\\
&+i\left[\frac{1}{2}\left(p_{z\uparrow,z\downarrow} + p_{z\downarrow,z\downarrow}\right) - p_{y\uparrow,z\downarrow}\right],\\
&\rho_{34} = \rho_{43}^{*} = \frac{1}{2}\left(p_{z\downarrow,z\uparrow} + p_{z\downarrow,z\downarrow}\right) - p_{z\downarrow,x\downarrow}\\
&+i\left[\frac{1}{2}\left(p_{z\downarrow,z\uparrow} + p_{z\downarrow,z\downarrow}\right) - p_{z\downarrow,y\uparrow}\right].
\end{aligned}
\tag{16.53}
$$

In view of these relations, we can express the density matrix elements in terms of probability distributions of spin projections as follows:

$$
\begin{aligned}
&\rho_{11} = p_{z\uparrow,z\uparrow}, \quad \rho_{22} = p_{z\uparrow,z\downarrow}, \quad \rho_{33} = p_{z\downarrow,z\uparrow}, \quad \rho_{44} = p_{z\downarrow,z\downarrow},\\
&\mathrm{Re}\rho_{12} = \mathrm{Re}\rho_{21} = p_{z\uparrow,x\uparrow} - \frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\uparrow,z\downarrow}\right),\\
&\mathrm{Re}\rho_{13} = \mathrm{Re}\rho_{31} = p_{x\uparrow,z\uparrow} - \frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\downarrow,z\uparrow}\right),\\
&\mathrm{Re}\rho_{14} = \mathrm{Re}\rho_{41} = \frac{1}{2}\left(p_{x\uparrow,x\uparrow} + p_{x\downarrow,x\downarrow} - p_{y\uparrow,y\uparrow} - p_{y\downarrow,y\downarrow}\right),\\
&\mathrm{Re}\rho_{23} = \mathrm{Re}\rho_{32} = \frac{1}{2}\left(p_{x\uparrow,x\uparrow} + p_{x\downarrow,x\downarrow} - p_{y\uparrow,y\downarrow} - p_{y\downarrow,y\uparrow}\right),\\
&\mathrm{Re}\rho_{24} = \mathrm{Re}\rho_{42} = \frac{1}{2}\left(p_{z\uparrow,z\downarrow} + p_{z\downarrow,z\downarrow}\right) - p_{x\downarrow,z\downarrow},\\
&\mathrm{Re}\rho_{34} = \mathrm{Re}\rho_{43} = \frac{1}{2}\left(p_{z\downarrow,z\uparrow} + p_{z\downarrow,z\downarrow}\right) - p_{z\downarrow,x\downarrow},\\
&\mathrm{Im}\rho_{12} = -\mathrm{Im}\rho_{21} = \frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\uparrow,z\downarrow}\right) - p_{z\uparrow,y\uparrow},\\
&\mathrm{Im}\rho_{13} = -\mathrm{Im}\rho_{31} = \frac{1}{2}\left(p_{z\uparrow,z\uparrow} + p_{z\uparrow,z\downarrow}\right) - p_{y\uparrow,z\uparrow},\\
&\mathrm{Im}\rho_{14} = -\mathrm{Im}\rho_{41} = \frac{1}{2}\left(p_{y\uparrow,x\downarrow} + p_{y\downarrow,x\uparrow} + p_{x\uparrow,y\downarrow} + p_{x\downarrow,y\uparrow} - 1\right),\\
&\mathrm{Im}\rho_{23} = -\mathrm{Im}\rho_{32} = \frac{1}{2}\left(p_{y\uparrow,x\downarrow} + p_{y\downarrow,x\uparrow} - p_{x\uparrow,y\downarrow} - p_{x\downarrow,y\uparrow}\right),\\
&\mathrm{Im}\rho_{24} = -\mathrm{Im}\rho_{42} = \frac{1}{2}\left(p_{z\uparrow,z\downarrow} + p_{z\downarrow,z\downarrow}\right) -\\
&\mathrm{Im}\rho_{34} = -\mathrm{Im}\rho_{43} = \frac{1}{2}\left(p_{z\downarrow,z\uparrow} + p_{z\downarrow,z\downarrow}\right) - p_{z\downarrow,y\uparrow},
\end{aligned}
\tag{16.54}
$$

These formulas provide the possibility to obtain the density matrix of two qubit states by measuring the above probabilities.

16.6 Conclusions

To conclude, we point out the main results discussed in our work.

We showed that the old problem of quantum physics, which was not solved during one century, namely, the problem of describing the states of systems in the nature can be solved, by means of conventional probability functions. On examples of usual systems, like the harmonic oscillator and qubits, we showed that, in addition to standard description of states employed in all textbooks on quantum mechanics, the states of harmonic oscillator and qubits can be described by conventional classical conditional probability distributions. These probability distributions, like tomographic probability distributions, for oscillator systems also determine usual density matrices and, for pure oscillator states like deformed oscillator, wave functions. Thus, all information on quantum states available in the tomographic probability distributions for systems like oscillators completely coincides with information contained in the wave function or density matrix, including the energy spectrum and the evolution associated with the dynamics described by the Schrödinger equation.

The probability distributions describing quantum states of oscillators and qubits contain information on the entanglement in the system associated with the superposition principle which, in turn, is related to superposition of wave functions of two-mode oscillators. The entangled quantum states provide the possibility to obtain new kinds of classical probability distributions called the entangled probability distributions, which have not been studied in conventional probability theory, but will be studied in the future publications. For spin systems, an explicit form of the probability representation, we presented here, provides the possibility to discuss the properties of Bell's states, which are entangled spin states, and generalize the known entangled qubit probability distribution to multi-qudit systems.

The problem of extending our approach to quantum field theory and obtaining the probability representation for the case of quantum field theory is the other aspect which will be considered in the future publications.

Thus, in our work on examples of the simplest quantum systems, like oscillators and spin-1/2 systems, we found that the classical description of states of usual classical oscillator can be extended to quantum oscillator by generalizing the classical Radon transform to the quantum Radon transform, that also needs to be deeper studied in mathematical physics.

The methods used in the consideration of probability representations of quantum states like the method of integrals of motion for systems with quadratic Hamiltonians [25, 26] can be used to obtain explicit expressions for the conditional probability distributions describing the system states; for example, all problems of charges moving in magnetic fields like the problem of Landau levels [27] and its development for a more complicated situation [28, 29]. Analogously, it is worth to try to formulate the influence of classical parameters on quantum phenomena studied in [30] and nonstationary Casimir effect studied in [31, 32] using the probability representation of quantum mechanics.

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16.7 References

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