Chapter 13

The quantum states of inverted and usual oscillators and particle with spin–1/2 states in probability representation of quantum mechanics

S. N. Belolipetskiy[1], V. N. Chernega[2], V. I. Grebenkin[1] and O. V. Man'ko[1,3]

[1]Special Educational Scientific Center, Bauman Moscow State Technical University, The 2nd Baumanskaya Street 5, Moscow 105005, Russia

[2]Institute of Managment and Digital Technologies, Department of Logistics and Transport System Managment, Russian University of Transport (MIIT), Obraztsova Street, 9/9, Moscow 127994

[3]Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia

13.1 Introduction

The main aim of this paper is to discuss the possibility to describe the quantum states of physical system in the framework of conventional quantum mechanics using standard probability distribution.

Evolution of coherent and some another states of harmonic oscillator for which at time $t = 0$ the frequency $\omega_{in} = 1$ and finally changes becoming $\omega(t > 0) = i\omega = i$ (so called inverted oscillator) is described by conventional probability distribution which is alternative of density operator in probability represention of quantum mechanics. The inverted oscillator was studied e.g. in [1, 2] in connection with relation to cosmological problems. Some of the inverted oscillator problems were also considered in [3–6]. The method of time-dependent intehrals of motion for systems like inverted oscillator is used to find the explicit form of the probability distributions [7] describes the states of the oscillator dynamics.

We consider this possibility onto simplest examples like spin–1/2 systems and free motion of a particle. We point out the canonical method to describe the states of quantum system is to associate the states either with complex wave function Dirac [8] or with the density operators [9, 10] acting in a Hilbert space [11]. Our goal is to construct the invertable map of the wave functions and density matrices onto conventional nonnegative probabilities obeying quantum evolution equations similar to equations used in classical statistical mechanics for describing the states of classical systems taking into account the fluctuations of positions and momenta of the particles. The probability representation of quantum mechanics was introduced for systems with continious variables like harmonic oscillator using quantum tomographic probability distribution of the oscillator position [12, 13] and for spin systems in [14–18].

In this paper we present the example of free motion coherent states in the tomographic probability distribution representation based on using the time–dependent integrals of motion linear in position and momentum operators [19–25]. The method of such integrals of motion was used in this works to construct coherent states of arbitrary quantum systems with the Hamiltonians quadratic in position and momenta operators. Quantum evolution equation for open system was considered [26–29]. In [30] it was shown that in classical mechanics can be introduced the Hermitian operators and the concepts of classical mechanics can be formulated using the inverse Wigner-Weyl transform of the classical probability distributions. In [31] the attempts to study the problems of complex biosystems composed of many subsystems and in [32] the attempts to model social process using the language of quantum mechanics, thermodynamics, quantum information and field theory were done. In [33] a review of classical probability representations of quantum states and observables is done and in [34] quantum nonlocality is discussed. New fundamental aspects of quantum mechanics based on groupoid approach are investigated in [35]. The problem of the probabilities used in quantum mechanics was discussed in [36] . The methods of star-product, tomography and probability representation of quantum mechanics were applied to different problems of quantum phenomena in [37–47]. The geometrical interpretation of spin state in probability representation with the help of Malevich's squares is done in [48, 49]. Applications of quantum tomographic approach to different kinds of experiments as well as theoretical researches were discussed in [50–54].

The paper is organized as follows. In Sec. 2 we consider the inverted oscillator and its evolution in probability representation. In Sec. 3 we consider spin–1/2 system (two–level atom). In Sec. 4 we consider the spin-1/2 states, spin projections as physical observables and superposition principle in probability representation of quantum mechanics. In Sec. 5 we consider Schrödinger equation in probability representation of quantum mechanics. In Sec. 6 we discuss new entropic inequalities. In Sec. 7 the quantum tomography of a free particle and its relation to the Wigner quasiprobability distribution (Wigner function [55]) is studied. An example of a coherent state of free particle is obtained as limit of oscillator coherent state for frequency equal to zero.

13.2 Inverted oscillator and its evolution in probability representation of quantum mechanics

The Hamiltonian of inverted oscillator with mass $m = 1$ and frequency $\omega^2 = -1$ has the form

$$\hat{H} = \frac{\hat{p}^2}{2} - \frac{\hat{q}^2}{2} \tag{13.1}$$

where $\hat{p} = -i\frac{\partial}{\partial x}$, $\hat{q} = x$ (we assume Planck constant $\hbar = 1$). The integrals of motion $\hat{q}_0(t)$ and $\hat{p}_0(t)$ which satisfy the conditions $\hat{q}_0(t=0) = \hat{q}$, $\hat{p}_0(t=0) = \hat{p}$ and the evolution equation $\frac{d\hat{q}_0(t)}{dt} = 0$, $\frac{d\hat{p}_0(t)}{dt} = 0$ or

$$\frac{\partial \hat{q}_0(t)}{\partial t} + i\left[\frac{\hat{p}^2}{2} - \frac{\hat{q}^2}{2}, \hat{q}_0(t)\right] = 0, \tag{13.2}$$

$$\frac{\partial \hat{p}_0(t)}{\partial t} + i\left[\frac{\hat{p}^2}{2} - \frac{\hat{q}^2}{2}, \hat{p}_0(t)\right] = 0, \tag{13.3}$$

have the form of time-dependent linear combination of operators $\hat{q}$ and $\hat{p}$, i.e.,

$$\hat{q}_o(\hat{q}, \hat{p}, t) = \hat{q}\cosh t - \hat{p}\sinh t, \tag{13.4}$$

$$\hat{p}_o(\hat{q}, \hat{p}, t) = -\hat{q}\sinh t + \hat{p}\cosh t. \tag{13.5}$$

The evolution of the density operator of any physical system state with time-independent Hamiltonian $\hat{H}$ is determined by unitary operator $\hat{U}(t) = \exp(-i\hat{H}t)$ using the expresion for the evolving density operator $\hat{\rho}(t)$ of the form

$$\hat{\rho}(t) = \hat{u}(t)\hat{\rho}(0)\hat{u}^\dagger(t) \tag{13.6}$$

which is integral of motion. The probability distribution $w(X|\mu, \nu, t)$, where X is position and μ, ν are parameters of the state with density operator (13.6) reads [7]

$$w_{\hat{\rho}}(X|\mu, \nu) = \mathrm{Tr}\hat{\rho}\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}) \tag{13.7}$$

In view of (13.6) one can express (13.7) in terms of expression

$$w_{\hat{\rho}}(X|\mu, \nu, t) = \mathrm{Tr}\hat{\rho}(0)\delta(X\hat{1} - \mu_H\hat{q} - \nu_H\hat{p}) \tag{13.8}$$

which is the probability distribution describing the initial state of the oscillator with parameters $\mu_H(t)$ and $\nu_H(t)$ corresponding to the Heisenberg position $\hat{q}_H(t) = \hat{u}^\dagger(t)\hat{q}\hat{u}(t)$ and momentum $\hat{p}_H(t) = \hat{u}^\dagger(t)\hat{p}\hat{u}(t)$ operators providing the equality

$$X\hat{1} - \mu_H(t)\hat{q} - \nu_H(t)\hat{p} = X\hat{1} - \mu\hat{q}_0(t) - \nu\hat{p}_0(t), \tag{13.9}$$

where we use the relations of integrals of motion $q_0(t)$ and $p_0(t)$ with the Heisenberg operators $\hat{q}_H(t) = \hat{q}_0(-t)$ and $\hat{p}_H(t) = \hat{p}_0(-t)$, i.e.

$$\mu_H = \mu\cosh t + \mu\sinh t \quad \text{and} \quad \nu_H = \mu\sinh t + \mu\cosh t. \tag{13.10}$$

Thus, if initial probability distribution at $t = o$ determines the coherent state of usual oscillator $\psi_\alpha(x,0)$, i. e.

$$w_0(X,\mu,\nu,t=0) = \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\exp\left(-\frac{(X-\bar{X}(0))^2}{\mu^2+\nu^2}\right). \tag{13.11}$$

where

$$\bar{X} = \sqrt{2}\,(\mu\mathrm{Re}\alpha + \nu\mathrm{Im}\alpha) \tag{13.12}$$

in potential of inverted oscillator it involves as normal distribution with dispersion

$$\sigma = \mu_H^2(t) + \nu_H^2(t)\,\mathrm{and}\,\bar{X}(t) = \sqrt{2}\,(\mu_H(t)\mathrm{Re}\alpha + \nu_H(t)\mathrm{Im}\alpha)\,. \tag{13.13}$$

13.3 State vector. Spin

Let us consider the spin-1/2 state. It is determined by complex vector which in Dirac form is [11]

$$|\psi\rangle = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad |\psi_1|^2 + |\psi_2|^2 = 1, \tag{13.14}$$

where $|\psi_1|^2$ is probability to find spin-projection on axis z equal to $m = +1/2$, and $|\psi_2|^2$ is probability to find spin projection on the axis z equal to $m = -1/2$.

If the quantum particle is interacting with the medium, then the additional fluctuations appear and the state of the particle is determined by the density matrix [9, 10]. For example the mixed state density matrix of harmonic oscillator is of the form

$$\rho(q,q') = \sum_k P_k\psi_k(q)\psi_k^*(q'), \quad \sum_k P_k = 1. \tag{13.15}$$

The spin-1/2 mixed state is determined by the density matrix which is a mixture

$$\begin{aligned}&\rho_{mm'} = \lambda_1(|\psi^{(1)}\rangle\langle\psi^{(1)}|)_{mm'} + \lambda_2(|\psi^{(2)}\rangle\langle\psi^{(2)}|)_{mm'} + \ldots,\\ &\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \ldots, \quad \lambda_1 + \lambda_2 + \ldots = 1.\end{aligned} \tag{13.16}$$

The pure state also can be determined by density matrix. For example, the density matrix of pure oscillator state is expressed through the scalar product of state vectors

$$\rho_\psi(q,q') = \psi(q)\psi^*(q').$$

For spin 1/2 state the density matrix of pure state is

$$\rho_{mm'}^\psi = |\psi\rangle\langle\psi| = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\begin{pmatrix}\psi_1^* & \psi_2^*\end{pmatrix} = \begin{pmatrix}\psi_1\psi_1^* & \psi_1\psi_2^*\\ \psi_2\psi_1^* & \psi_2\psi_2^*\end{pmatrix}, \tag{13.17}$$

where $m = \pm 1/2, \quad m' = \pm 1/2$.

In classical mechanics the observables are the functions of coordinate q and momentum p. For example, the energy of oscillator is of the form

$$E(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{13.18}$$

The observables in quantum mechanics are hermitian operators. Energy operator (the Hamiltonian) of spin 1/2 is hermitian operator, which is determined by matrix

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \quad H^\dagger = H. \tag{13.19}$$

The eigenvalues of observables are real numbers. For example, the observable which is spin projection on axis z is of the form $\hat{S}_Z = \frac{\hbar}{2}\sigma_z$, the operator of spin projection on axis x is $\hat{S}_x = \frac{\hbar}{2}\sigma_x$, operator of spin projection on axis y is $\hat{S}_y = \frac{\hbar}{2}\sigma_y$, where matrices σ_z, σ_x, σ_y are the Pauli matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{13.20}$$

13.4 Probability representation for the spin–1/2 state

Let us consider the state with the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{13.21}$$

and solve the following problem: What is the probability that the measurement of spin projection on axis z is equal to $m = +1/2$? The answer is: this probability p_3 is equal to ρ_{11},

$$p_3 = \mathrm{Tr}\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \rho_{11}. \tag{13.22}$$

What is the probability to find under the measurement of the spin projection on axis x the value equal to $m = +1/2$ in the state (13.21)? This probability p_1 is $\frac{1}{2} + \frac{\rho_{12}+\rho_{21}}{2}$. So one has

$$p_1 = \mathrm{Tr}\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\right) = \frac{1}{2} + \frac{\rho_{12}+\rho_{21}}{2} = \frac{1}{2} + \mathrm{Re}\rho_{12}. \tag{13.23}$$

What is the probability to find under the measurement of the spin projection on axis y the value equal to $m = +1/2$ in the state (13.21)? This probability p_2 is $\frac{1}{2} - \mathrm{Im}\rho_{12}$, it is equal

$$p_2 = \mathrm{Tr}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix} = \frac{1}{2} + \mathrm{Im}\rho_{21} = \frac{1}{2} - \mathrm{Im}\rho_{12}. \tag{13.24}$$

We obtain, that the matrix ρ (13.21) is of the form

$$\rho = \begin{pmatrix} p_3 & p_1 - 1/2 - i(p_2 - 1/2) \\ p_1 - 1/2 + i(p_2 - 1/2) & 1 - p_3 \end{pmatrix}. \tag{13.25}$$

Any observable (13.19) can be written in the form

$$H = \begin{pmatrix} z_1 & x - iy \\ x + iy & z_2 \end{pmatrix}. \tag{13.26}$$

The statistical properties of the observable are determined in quantum mechanics through mean values, dispersions and higher momenta. For example, the mean values are

$$\langle H\rangle = \mathrm{Tr}\,(\rho H) = \mathrm{Tr}\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\right),$$

so

$$\langle H\rangle = \rho_{11}H_{11} + \rho_{12}H_{21} + \rho_{21}H_{12} + \rho_{22}H_{22}.$$

Our aim is to express these quantum quantities through the analogs of three classical random variables

$$\vec{X} = \begin{pmatrix} x \\ -x \end{pmatrix}, \tag{13.27}$$

$$\vec{Y} = \begin{pmatrix} y \\ -y \end{pmatrix}, \tag{13.28}$$

$$\vec{Z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{13.29}$$

and probabilities p_1, p_2, p_3. Let us show the expression for mean value of the observable $\langle H\rangle$ as a result of a game with three classical coins. Let the first coin has the probability distribution of the form

$$\vec{p}_1 = \begin{pmatrix} p_1 \\ 1-p_1 \end{pmatrix}, \tag{13.30}$$

then $\pm$ or "heads" and "tails" are connected with random variable $\vec{X}$ (13.27). The second coin with probability distribution

$$\vec{p}_2 = \begin{pmatrix} p_2 \\ 1-p_2 \end{pmatrix}, \tag{13.31}$$

is connected with random variable $\vec{Y}$ (13.28), and the third one with probability distribution

$$\vec{p}_3 = \begin{pmatrix} p_3 \\ 1-p_3 \end{pmatrix}, \tag{13.32}$$

is connected with random variable

$$\vec{Z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{13.33}$$

The numbers x, y, z_1, z_2 can be arranged in a matrix form, so one has for H

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} z_1 & x-iy \\ x+iy & z_2 \end{pmatrix}. \tag{13.34}$$

Then the quantum mean value of observable H, namely $\langle H\rangle = \mathrm{Tr}(\rho H)$, is

$$\begin{aligned}\langle H\rangle &= \langle\vec{X}\rangle + \langle\vec{Y}\rangle + \langle\vec{Z}\rangle = [p_1x + (1-p_1)(-x)] + [p_2y + (1-p_2)(-y)] \\ &\quad + [p_3z_1 + (1-p_3)z_2].\end{aligned}$$

One has for the random variable $\vec{X}$ (13.27), where when a coin lands on "heads", the payoff is $+x$, and in case of "tails", the loss is $(-x)$, then the average income $\langle\vec{X}\rangle$ according to the classical theory of probability is

$$p_1(x) + (1-p_1)(-x) = \langle\vec{X}\rangle.$$

Analogously, the same one has for $\langle\vec{Y}\rangle$ and $\langle\vec{Z}\rangle$.

We can conclude that the state of the particle with the spin–1/2 can be interpreted in quantum mechanics as a set of probabilities given by vectors

$$\vec{P} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \text{and} \quad (1-\vec{P}) = \begin{pmatrix} 1-p_1 \\ 1-p_2 \\ 1-p_3 \end{pmatrix}, \tag{13.35}$$

and observables, for example, (13.19), can be interpreted as a set of random variables $\vec{X}$, $\vec{Y}$, $\vec{Z}$ in the game with classical coins, which are determined by probabilities p_1, p_2, p_3, $1-p_1$, $1-p_2$, $1-p_3$. So, we show on the example of the spin $s = 1/2$, that quantum states can be determined by probability distributions for classical coins, and quantum observables can be determined by corresponding classical, random variables. This approach can be generalized to other spin systems. The approach can be illustrated by paraphrasing Einstein's discussion with Bohr, namely, the statement "God does not play dice - God plays with coins" [56].

Expressions for higher momenta can be obtained, for example, dispersion of the observable

$$\sigma_H = \langle H^2\rangle - (\langle H\rangle)^2 = \mathrm{Tr}\left(\rho H^2\right) - \left(\mathrm{Tr}\left(\rho H\right)\right)^2$$

is expressed through the probabilities p_1, p_2, p_3 and values of random variables $\vec{X}$, $\vec{Y}$, $\vec{Z}$.

The quantumness of a state with a density matrix ρ imposes a condition on the probabilities in the game with classical coins, namely, the density matrix must have only non-negative eigenvalues. We get the inequality

$$\det\rho = 1/4 - (p_1 - 1/2)^2 - (p_2 - 1/2)^2 - (p_3 - 1/2)^2 \geq 0. \tag{13.36}$$

This inequality (13.36) can be violated in the case of the game with classical coins, that is, it does not hold for all probabilities $0 \leq p_1, p_2, p_3 \leq 1$ [57]. For pure quantum states, inequality turns into equality, that is, if $\rho_\psi = |\psi\rangle\langle\psi|$, then the probabilities p_1, p_2, p_3 lie on the surface of a sphere of radius $R = 1/2$ given by the equation

$$(p_1 - 1/2)^2 + (p_2 - 1/2)^2 + (p_3 - 1/2)^2 = 1/4. \tag{13.37}$$

We can express the state vector through three probabilities p_1, p_2, p_3 also in the case of pure state $\rho_\psi = |\psi\rangle\langle\psi|$. The state vector

$$|\psi\rangle = \begin{pmatrix} \sqrt{p_3} \\ \frac{(p_1 - 1/2)}{\sqrt{p_3}} + \frac{i(p_2 - 1/2)}{\sqrt{p_3}} \end{pmatrix} \tag{13.38}$$

gives the expression for density matrix

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} p_3 & p_1 - 1/2 - i(p_2 - 1/2) \\ p_1 - 1/2 + i(p_2 - 1/2) & 1 - p_3 \end{pmatrix}. \tag{13.39}$$

This expression implies the formulation of the principle of superposition as a rule for nonlinear addition of probabilities.

Given two orthogonal, pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, determined by the probabilities p_1, p_2, p_3 (first state) and $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ (second state), then their superposition has the form

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle,$$

where

$$c_1 = \sqrt{\Pi_3}, \quad c_2 = \frac{\Pi_1 - 1/2}{\sqrt{\Pi_3}} + i\frac{\Pi_2 - 1/2}{\sqrt{\Pi_3}},$$

and numbers Π_1, Π_2, Π_3 are formally probabilities lying on the surface of the same sphere as the probabilities p_1, p_2, p_3, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$. The superposition $|\psi\rangle$ is given by a set of probabilities P_1, P_2, P_3, which are expressed in terms of probabilities p_1, p_2, p_3, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and numbers Π_1, Π_2, Π_3 as follows

$$\begin{aligned}
&P_3 = \Pi_3 p_3 + (1-\Pi_3)\mathcal{P}_3 + 2\sqrt{p_3\mathcal{P}_3}\,(\Pi_1 - 1/2)\,,\\
&P_1 - 1/2 = \Pi_3(p_1 - 1/2) + (\mathcal{P}_1 - 1/2)(1-\Pi_3)\\
&+[(\Pi_1 - 1/2)(p_1 - 1/2) + (\Pi_2 - 1/2)(p_2 - 1/2)]\sqrt{\frac{\mathcal{P}_3}{p_3}}\\
&+[(\Pi_1 - 1/2)(\mathcal{P}_1 - 1/2) - (\Pi_2 - 1/2)(\mathcal{P}_2 - 1/2)]\sqrt{\frac{p_3}{\mathcal{P}_3}},\\
&P_2 - 1/2 = (p_2 - 1/2)\Pi_3 + (\mathcal{P}_2 - 1/2)(1-\Pi_3)\\
&+\sqrt{\frac{\mathcal{P}_3}{p_3}}\,[(\Pi_1 - 1/2)(p_2 - 1/2) - (\Pi_2 - 1/2)(p_1 - 1/2)]\\
&+\sqrt{\frac{p_3}{\mathcal{P}_3}}\,[(\Pi_2 - 1/2)(\mathcal{P}_1 - 1/2) + (\Pi_1 - 1/2)(\mathcal{P}_2 - 1/2)]\,.
\end{aligned}$$

13.5 Schrödinger equation

In quantum mechanics the energy of the system has the quantized values. The energy levels are defined by the solutions of the Schrödinger equation

$$\hat{H}|\psi\rangle = E|\psi\rangle \tag{13.40}$$

In the case of the particle with spin $s = 1/2$ (qubit) the equality is of the form

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{13.41}$$

then we obtain

$$\begin{aligned}
H_{11}\psi_1 + H_{12}\psi_2 &= E\psi_1,\\
H_{21}\psi_1 + H_{22}\psi_2 &= E\psi_2.
\end{aligned}$$

The solution is given by the secular equation

$$\det\begin{pmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{pmatrix} = 0. \tag{13.42}$$

We get the quadratic equation

$$H_{11}H_{22} - EH_{22} - EH_{11} + E^2 - H_{21}H_{12} = 0,$$

with the solution of the form

$$E_{1,2} = \frac{H_{11} + H_{22}}{2} \pm \sqrt{\frac{(H_{11} + H_{22})^2}{4} - H_{11}H_{22} + H_{21}H_{12}}. \qquad (13.43)$$

The Schrödinger equation specifying the energy levels of a particle with spin $s = 1/2$ can be written for the density matrix. We multiply the Schrödinger equation by the bra-vector $\langle\psi|$ on the right

$$\hat{H}|\psi\rangle\langle\psi| = E|\psi\rangle\langle\psi|$$

and we understand the vector as a rectangular matrix

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\psi_1^* \quad \psi_2^*) = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\psi_1^* \quad \psi_2^*). \qquad (13.44)$$

Introducing

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\psi_1^* \quad \psi_2^*) = \begin{pmatrix} \psi_1\psi_1^* & \psi_1\psi_2^* \\ \psi_2\psi_1^* & \psi_2\psi_2^* \end{pmatrix}, \qquad (13.45)$$

we get

$$\rho_{11} = \psi_1\psi_1^*,\ \rho_{12} = \psi_1\psi_2^*,\ \rho_{21} = \psi_2\psi_1^*,\ \rho_{22} = \psi_2\psi_2^*. \qquad (13.46)$$

Using (13.45), (13.46)

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = E \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21}| & \rho_{22} \end{pmatrix},$$

we obtain four equations

$$H_{11}\rho_{11} + H_{12}\rho_{21} = E\rho_{11}, \quad H_{11}\rho_{12} + H_{22}\rho_{22} = E\rho_{12},$$

$$H_{21}\rho_{11} + H_{22}\rho_{21} = E\rho_{21}, \quad H_{21}\rho_{12} + H_{22}\rho_{22} = E\rho_{22}.$$

It can be verified that these relations are obtained from the matrix equation for the eigenvectors

$$\begin{pmatrix} H_{11} & 0 & H_{12} & 0 \\ 0 & H_{11} & 0 & H_{12} \\ H_{21} & 0 & H_{22} & 0 \\ 0 & H_{21} & 0 & H_{22} \end{pmatrix} \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} = E \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix}. \qquad (13.47)$$

Let us write the expression for density matrix in probability representation (13.25), where p_1, p_2, p_3 are spin projections $m = +1/2$ on axis x, y, z. Let us rewrite the resulting matrix equation, which determines the energy levels for the spin $s = 1/2$ in terms of the matrix equation for the probability vector

$$\begin{pmatrix} H_{11} & 0 & H_{12} & 0 \\ 0 & H_{11} & 0 & H_{12} \\ H_{21} & 0 & H_{22} & 0 \\ 0 & H_{21} & 0 & H_{22} \end{pmatrix} \begin{pmatrix} p_3 \\ p_1 - 1/2 - i(p_2 - 1/2) \\ p_1 - 1/2 + i(p_2 - 1/2) \\ 1 - p_3 \end{pmatrix}$$

$$= E \begin{pmatrix} p_3 \\ p_1 - 1/2 - i(p_2 - 1/2) \\ p_1 - 1/2 + i(p_2 - 1/2) \\ 1 - p_3 \end{pmatrix}. \qquad (13.48)$$

We have written down the Schrödinger equation for the energy levels of a particle with spin $s = 1/2$ and Hamiltonian H, as an equation for the probabilities p_1, p_2, p_3 defining a quantum state. Introducing the matrix $\mathcal{H} = H \otimes 1$ as the tensor product of the Hamiltonian matrix and the identity matrix, as well as the four dimensional probability vector

$$|\mathcal{P}\rangle = \begin{pmatrix} p_3 \\ p_1 - 1/2 - i(p_2 - 1/2) \\ p_1 - 1/2 + i(p_2 - 1/2) \\ 1 - p_3 \end{pmatrix}, \tag{13.49}$$

we can write the Schrödinger equation for the energy levels of a qubit in the form

$$\mathcal{H} \cdot |\mathcal{P}\rangle = E|\mathcal{P}\rangle. \tag{13.50}$$

This form is valid for any values of the energy levels of particles with any spin (qudits) [57, 58].

13.6 Entropic inequalities

Let us introduce a new entropy for the states of a qubit, characterizing this state, using the Shannon entropy

$$\mathbf{H} = -p_1 \ln p_1 - (1-p_1)\ln(1-p_1) - -p_2 \ln p_2 - (1-p_2)\ln(1-p_2) - p_3 \ln p_3 - (1-p_3)\ln(1-p_3). \tag{13.51}$$

This new entropy characterizes the randomness of the distribution of spin projections along three mutually perpendicular directions in a given state of a particle with spin $s = 1/2$ with density matrix ρ. By measuring the probabilities p_1, p_2, p_3, we can obtain experimental values of this entropy. The properties of Shannon's entropy are known, they are given by inequalities for the relative entropy. Suppose we have given two probability distributions, similar to (13.30), (13.31), then one has

$$p_1 \ln\left(\frac{p_1}{p_2}\right) + (1 - p_1)\ln\left(\frac{1 - p_1}{1 - p_2}\right) \geq 0. \tag{13.52}$$

This new relationship can be verified experimentally by measuring the projection of the spin onto two perpendicular directions. Using these relations, you can write new entropy equalities and inequalities. Similar entropy inequalities can be obtained for arbitrary qudits.

13.7 Free motion coherent states in probability representation

We consider a free particle with mass $M = 1$ and assume Planck constant $\hbar = 1$. The Hamiltonian of the free particle

$$\hat{H} = \frac{\hat{p}^2}{2}. \tag{13.53}$$

This Hamiltonian is limit case of oscillator Hamiltonian for frequency $\omega - 0$. One can check that there are two integrals of motion in Schrödinger representation [22]

$$\hat{q}_0(t) = \hat{q} - \hat{p}t, \quad \hat{p}_0(t) = \hat{p}, \tag{13.54}$$

where position and momentum operators $\hat{q}$ and $\hat{p}$ satisfy the commutation relation $[\hat{q}, \hat{p}] = i\hat{1}$. One can check that the operators (13.54) satisfy the equation for integrals of motion [22, 59] for free particle

$$\frac{d}{dt}\hat{q}_0(t) = \frac{\partial}{\partial t}\hat{q}_0(t) + i\left[\frac{\hat{p}^2}{2}, q_0(t)\right] = 0, \quad \frac{d}{dt}\hat{p}_0(t) = 0. \tag{13.55}$$

The nonhermitian operators

$$\hat{A}(t) = \frac{\hat{q}_0(t) + i\hat{p}_0(t)}{\sqrt{2}}, \quad \hat{A}^\dagger(t) = \frac{\hat{q}_0(t) - i\hat{p}_0(t)}{\sqrt{2}} \tag{13.56}$$

(annihilation and creation operators) are integrals of motion and they satisfy boson commutation relations

$$[\hat{A}(t), \hat{A}^\dagger(t)] = \hat{1}. \tag{13.57}$$

The standart method to construct coherent states using the operators (13.56) [60, 61] is to find the ground–like state $|0\rangle$ satisfying the equation

$$\hat{A}|0, t\rangle = 0, \tag{13.58}$$

which has the normalized wave function in position representation

$$\psi_0(x, t) = \frac{1}{(\pi)^{1/4}}\frac{1}{1 + it}\exp\left(-\frac{x^2}{2(1 + it)}\right). \tag{13.59}$$

Acting on this function by displacement operator $\hat{D}(\alpha) = \exp\left(\alpha\hat{A}^\dagger(t) - \alpha^*\hat{A}(t)\right)$. We obtain coherent states $|\alpha, t\rangle$ satisfying the equation

$$\hat{A}(t)|\alpha, t\rangle = \alpha|\alpha, t\rangle. \tag{13.60}$$

The normalized wave function of the coherent states in position representation reads

$$\psi_\alpha(x, t) = \frac{e^{-\frac{|\alpha|^2}{2}}}{(\pi)^{1/4}(1 + it)}\exp\left(-\frac{x^2}{2(1 + it)} + \frac{\sqrt{2}x\alpha}{1 + it} + \frac{\alpha^2(it - 1)}{2(it + 1)}\right). \tag{13.61}$$

This function is the Gaussian eigenfunction of the integral of motion $\hat{A}(t)$ (13.56) with complex eigenvalue α.

The symplectic tomogram of the coherent states (13.61) is given by the fractional Fourier transform of the wave function [62]

$$w_\alpha(X|\mu, \nu, t) = \frac{1}{2\pi|\nu|}\left|\int \psi_\alpha(y, t)\exp\left(\frac{i\mu y^2}{2\nu} - \frac{iXy}{\nu}\right)dy\right|^2. \tag{13.62}$$

The tomogram has the following physical meaning. For $\mu = 1$, $\nu = 0$ the tomographic probability distribution (13.62) is the probability density of the position $X \to q$ for $\mu = 0$,

$\nu = 1$ the probability distribution is the probability density $X \to p$. The tomogram has an explicit form

$$w_\alpha(X|\mu,\nu,t) = \frac{1}{\sqrt{\pi(\mu^2+(\nu+\mu t)^2}} \exp\left(-\frac{(X-\bar{X}_\alpha)^2}{\mu^2+(\nu+\mu t)^2}\right), \qquad (13.63)$$

where the mean value of quadrature X is

$$\bar{X}_\alpha = \sqrt{2}\,(\mu \mathrm{Re}(\alpha) + (\nu+\mu t)\mathrm{Im}(\alpha))\,. \qquad (13.64)$$

This is the Gaussian probability distribution of random position X of free particle which determines the density matrix and the Wigner function of the coherent state of free particle

$$W_\alpha = \frac{1}{2\pi}\int w_\alpha(X|\mu,\nu,t)\exp\left(i(X-\mu q-\nu p)\right). \qquad (13.65)$$

The Wigner function is the normal probability distribution of the variables q and p with $(\delta q)^2 = \frac{1}{2}(1+t^2)$, $(\delta p)^2 = \frac{1}{2}$ and mean values $\bar{q} = \frac{1}{\sqrt{2}}(\mathrm{Re}(\alpha) + t\mathrm{Im}(\alpha))$, $\bar{p} = \frac{1}{\sqrt{2}}\mathrm{Im}(\alpha)$.

13.8 Conclusions

To conclude we point out the main results of our work. We constructed the probability distribution of the example of quantum inverted oscillator evolution using the initial state of the oscillator which is the usual coherent state. The obtained probability distribution describing the time-dependent density operator of inverted oscillator is Gaussian distribution with time-dependent mean value and dispersion. The obtained result can be extended to the cases of other systems with Hamiltonians described by quadratic forms of position and momentum operators like systems of charges moving in magnetic fields of solenoids which we consider in future publication.

Also we demonstrate that the spin–1/2 states can be described by three probabilities $0 \le p_1, p_2, p_3 \le 1$ which are probabilities to have spin projections $m = +1/2$ onto three perpendicular directions in the space. These probabilities determine the density matrix of arbitrary spin–1/2 states both pure and mixed states.

For quantum free particle we constructed the integrals of motion which have the physical meaning of the initial position and initial momentum in the phase space of mean values of this observables, determining trajectory of the free particle in the phase space. Using these integrals of motion we constructed the coherent states of the free particle and presented the probability distribution of the position of the particle determining the state Wigner function and density matrix in position representation using Radon transform [63] of the state tomogram. The state tomogram is explicitly found in the form of the Gaussian distribution function. The obtained results can be extended to other spin systems and to other systems with the Hamiltonians quadratic in position and momentum like charge particle in electric and magnetic field.

We made the review of the new formalism of quantum mechanics, in which the states of quantum systems are specified by probability distributions, as in classical statistical physics. In addition, the Schrödinger equation for energy levels was written in the form of a new equation for the probability. New entropy inequalities were obtained for the probabilities, which can be experimentally verified in experiments with spin $s = 1/2$

particle. The probabilities we have introduced are used in ordinary life as probabilities that determine the behavior of ordinary coins in the game "heads-tails". Speaking about the basic statement of quantum mechanics, we can, following Einstein, say "God does not play dice, God plays with coins." Due to this approach, about 100 years after the beginning of the development of quantum mechanics, it was possible to show that its abstract formulations from the point of view of the usual classical mechanics, including the concept of the wave function, density matrix, Hilbert space of states, can be explained intuitively by simple concepts of classical statistical mechanics. Namely, the wave function and the density operator can be replaced by the old well-known concepts - probability distributions, and the equations for the wave function and the density matrix can be replaced by the familiar linear equations for the probability distribution functions.

13.9 References

[1] F. Ullinger, M. Zimmermann, W. P. Schleich, The logarithmic phase singularity in the inverted harmonic oscillator, AVS Quantum Sci. **4(2)**, 024402 (2022).

[2] C. Subramanyan, S. S. Hegde, S. Vishveshwara, B. Bradlyn, Physics of the Inverted Harmonic Oscillator: From the lowest Landau level to event horizons, Annals Phys. **435**, 168470 (2021).

[3] W. Wöger, H. King, R. J. Glauber, J. W. Haus, Spontaneous generation of coherent optical beats, Phys. Rev. A **34**, 4859 (1986).

[4] A. Vourdas, F. Bishop, Quantum systems at negative temperatures: A holomorphic approach based on coherent states, J. Phys. A: Math. and Gen. **31(42)**, 8563 (1998).

[5] C. Yuce, A. Kilic, A. Coruh, Inverted oscillator, Phys. Scr. **74(1)**, 114 (2006).

[6] C. Yuce, Quantum inverted harmonic potential, Phys. Scr. **96(10)**, 105006 (2021).

[7] O. V. Man'ko, V. I. Man'ko, Inverted oscillator quantum states in the probability representation, Entropy **25(2)**, 217 (13p.) (2023).

[8] E. Schrödinger, Quantisierung als Eigenwertproblem (Erste Mitteilung), Ann. Phys. **79**, 361 (1926).

[9] L. Landau, Das Dämpfungsproblem in der Wellenmechanik, Z. Phys. **45**, 430 (1927).

[10] J. von Neumann, Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik, Nach. Ges. Wiss. Göttingen **11**, 245 (1927).

[11] P. A.M. Dirac, The Principles of Quantum Mechanics, 1958 (Clarendon Press: Oxford)

[12] S. Mancini, V. I. Man'ko, P. Tombesi, Symplectic Tomography as Classical Approach to Quantum Systems, Phys. Lett. A **213**, 1 (1996).

[13] O. V. Man'ko, V. I. Man'ko, Quantum States in Probability Representation and Tomography, J. Russ. Las. Res. **18**, 407 (1997).

[14] V. V. Dodonov, V. I. Man'ko, Positive Distribution Description for Spin States, Phys. Lett. A **239** 335 (1997).

[15] V. I. Man'ko, O. V. Man'ko, Spin State Tomography, J. Exp. Theor. Phys. **85**, 430 (1997).

[16] O.V. Man'ko, Tomography of Spin States and Classical Formulation of Quantum Mechanics, in: B. Gruber and M. Ramek (Eds.), *Proceedings of the International Conference "Symmetries in Science X" (Bregenz, Austria, 1997)*, 1998, Plenum Press, New York, p. 207

[17] J. P. Amiet, S. Weigert, Coherent states and the reconstruction of pure spin states, J. Opt. B: Quantum Semiclass. Opt., **1**, L5 - L8 (1999).
[18] G. M. D'Ariano, L. Maccone, M. Paini, Spin Tomography, J. Opt. B: Quantum Semiclass. Opt. **5**, 77 (2003).
[19] I. A. Malkin, V.I. Man'ko, Coherent states and excitation of N-dimensional nonstationary forced oscillator, Phys. Lett. A **32(4)**, 243 (1970).
[20] I. A. Malkin, V. I. Man'ko, D. Trifonov, Coherent States and Transition Probabilities in a Time-Dependent Electromagnetic Field, Phys. Rev. D **2**, 1371 (1970).
[21] V.V. Dodonov, I.A. Malkin, V.I. Man'ko, Even and odd coherent states and excitations of a singular oscillator, Physica **72(3)**, 597(1974).
[22] V. V. Dodonov, V. I. Man'ko, Invariants and correlated states of nonstationary quantum systems, in *Invariants and the Evolution of Nonstationary Quantum Systems*, *Proc. Lebedev Physics Institute, vol. 183 (Nova Science, Commack, N.Y.)* pp. 103–261 (1989).
[23] V. V. Dodonov, O. V. Man'ko, V. I. Manko, Time-dependent oscillator with Kronig-Penney excitation, Phys. Lett. A **175**, 1 (1993).
[24] V. N. Chernega, O. V. Man'ko, A Damped Oscillator with a δ–Kicked Frequency in the Probability Representation of Quantum Mechanics, J. Russ. Laser Res. **39**, 411 (2018).
[25] S. V. Kuznetsov, O. V. Man'ko, N. V. Tcherniega, J. Opt. B: Quantum Semiclas.l Opt., Photon distribution function, tomograms and entanglement in stimulated Raman scattering, J. Opt. B: Quantum Semiclass. Opt. **5**, 5503 (2003).
[26] Iu. A. Nosal', A. E. Teretenkov, Exact Dynamics of Moments and Correlation Functions for GKSL Fermionic Equations of Poisson Type, Math. Notes **108**, 911 (2020).
[27] A. E. Teretenkov, Irreversible Quantum Evolution with q Quadratic Generator: Review, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **22(4)**,1930001, 27 pp. (2019).
[28] S. Mancini, O. V. Man'ko, V. I. Man'ko, P. Tombesi, The Pauli equation for Probability Distributions, J. Phys. A: Math. Gen. **34**, 3461 (2001).
[29] Ya. A. Korennoy, Gauge-independent Husimi Functions of Charged Quantum Particles in the Electro-magnetic Field. arXiv:1806.06443 [quant-ph] (2018)
[30] P. W. Claeyes, A. Polkovnikov, Quantum eigenstates from classical Gibbs distributions, SciPost Phys. **10**, 014 (42pp.) (2021).
[31] A. Khrennikov, Born's formula from statistical mechanics of classical fields and theory of hitting times, Physica A: Statistical Mechanics and its Applications, **393**, 207 (2014).
[32] A. Khrennikov, *Social Laser*, Jenny Stanford Publ. 280 pp. (2020)
[33] A. Khrennikov A, A. Alodjants, Classical (Local and Contextual) Probability Model for Bohm–Bell Type Experiments: No-Signaling as Independence of Random Variables, Entropy **21**, 157 (2019).
[34] A. Khrennikov, Is the Devil in h? Entropy (Basel) **19;23(5)**, 632 (2021).
[35] F. M. Ciaglia, F. Di Cosmo, A. Ibort, G. Marmo, Evolution of Classical and Quantum States in the Groupoid Picture of Quantum Mechanics, Entropy 22(11), 1292 (2020).
[36] Y. V. Przhiyalkovskiy, Continuous Measurements in Probability Representation of Quantum Mechanics, Proc. Steklov Inst. Math. **313**, 193(2021).
[37] Elze H-T, Gambarotta G and Vallone F General Linear Dynamics—Quantum, Classical or Hybrid, J. Phys: Conf. Ser. **306**, 012010 (2011).

[38] M. A. de Gosson, Quantum Harmonic Analysis of the Density Matrix, Quanta **7**, 74 (2018).

[39] M. A. Gosson, C. de Gosson, Symplectic Polar Duality, Quantum Blobs, and Generalized Gaussians, Symmetry, **14(9)**,1890 (2022)

[40] M. A. Gosson, Symplectic Radon Transform and the Metaplectic Representation, Entropy, **24(6)**, 761 (2022)

[41] J. Foukzon, A. A. Potapov, E. Menkova, S. A. Podosenov, A New Quantum-Mechanical Formalism Based on the Probability Representation of Quantum States, viXra:1612.0298. (2016)

[42] V. N. Chernega, S. N. Belolipetskiy, O. V. Man'ko, V. I. Man'ko, Probability Representation of Quantum Mechanics and Star-Product Quantization, J. of Physics: Conf. Ser. **1348**, 012101 (2019).

[43] P. Facchi, M. Ligabó, Classical and quantum aspects of tomography in: M. Asorey, J. Clemente-Gallardo, E. Martínez, J. F. Cariñena (Ed.) *Proceedings of XVIII International Fall Workshop on Geometry and Physics, Benasque, Spain, 2009*, AIP Conference Proceedings **1260**, 3 (2010).

[44] C. Stornaiolo, Emergent classical universes from initial quantum states in a tomographical description, Int. J. Geom. Meth. in Modern Phys., **17(11)**, 2050167 (2020)

[45] C. Stornaiolo, Tomographic Represention of Quantum and Classical Cosmology, in: C. Moreno González, J. Madriz Aguilar, L. Reyes Barrera (Ed.), *Accelerated Cosmic Expansion. Proceedings of the Fourth International Meeting on Gravitation and Cosmology*, Astrophysics and Space Science Proceedings, **38**, 1155-1157 (2014) Springer

[46] M. R. Bazrafkan, E. Nahvifard, Stationary perturbation theory in the probability representation of quantum mechanics, J. Russ. Laser Res., **30**, 392——(2009).

[47] V. S. Filinov, G. Schubert, P. Levashov, M. Bonitz, H. Fehske, V. E. Fortov, A. V. Filinov, Center-of-mass tomographic approach to quantum dynamics, Phys. Lett. A **372** , 5064 (2008).

[48] V. N. Chernega, O. V. Man'ko, V. I. Man'ko, Triangle Geometry of the Qubit State in the Probability Representation Expressed in Terms of the Triada of Malevich's Squares, J. Rus. Laser Res. **38**, N 2, 141 (2017).

[49] V. N. Chernega, O. V. Man'ko, V. I. Man'ko, Triangle Geometry for Qutrit States in the Probability Representation, J. Rus. Laser Res. **38**, N 5, 416 (2017).

[50] G. P. Miroshnichenko, CQED Quantum Tomography of a Microwave Range, arXiv:1510.03155 [quant-ph] (2015)

[51] B. Koczor, R. Zeier, S. J. Glaser, Continuous Phase-Space Representations for Finite-Dimensional Quantum States and their Tomography, Phys. Rev. A, **101**, 022318 (2020)

[52] E. Toninell, B. Ndagano, A. Valles, A. Forbes, Concepts in quantum state tomography and classical implementation with intense light: a tutorial, Advances in Optics and Photonics, **11(1)**, 67 (2019).

[53] A. M. Almarashi, G. A. Abd-Elmougod, M.Z. Raqab, Quantum Extropy and Statistical Properties of the Radiation Field for Photonic Binomial and Even Binomial Distributions, J. Russ. Laser Res. **41**, 334 (2020).

[54] R. C. C.Leon, C. H. Yang, J. C. C. Hwang, J. C. Lemyre, T. Tanttu, W. Huang, J. H. Huang, F. E. Hudson, K. M. Itoh, A. Laucht, M. Pioro-Ladriére, A. Saraiva, A. S. Dzurak, Bell-state tomography in a silicon many-electron artificial molecule, Nat. Commun. **12**, 3228 (2021).

[55] E. Wigner, On the Quantum Correction For Thermodynamic Equilibrium, Phys. Rev. **40**, 749(1932).
[56] V. N. Chernega, O. V. Man'ko, V. I. Man'ko, Probability Representation of Quantum States as a Renaissance of Hidden Variables— God Plays Coins, J. Rus. Laser Res. **40**, N 2, 107 (2019).
[57] V. N. Chernega, O. V. Man'ko, V. I. Man'ko, Triangle Geometry for Qutrit States in the Probability Representation, J. Rus. Laser Res. **38**, N 4, 324 (2017).
[58] V. N. Chernega, O. V. Man'ko, V. I. Man'ko, Qubit state vector in probability representation of quantum mechanics, AIP **2362**, 549 (20 p.) (2021) in *Conference Proceedings MIPT-QUANT, 2020*
[59] L. D. Landau, E. M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, 3rd ed.; Elsevier, UK, 689 pp. (1981)
[60] R. J. Glauber, Coherent and Incoherent States of the Radiation Field, Phys. Rev. **131**, 2766(1963).
[61] E. C. G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. **10** 277 (1963).
[62] V. I. Man'ko, R. V. Mendes, Non-Commutative Time-Frequency Tomography, Phys. Lett. A **263**, 53 (1999).
[63] J. Radon, Uber die Bestimmung von Funktionen durch ihre Integralwerte Langs Gewisser Mannigfaltigkeiten, Ber. Verh. Sächs. Akad. Wiss. Leipzig **69**, 262(1917).

Cite this work as:

S. N. Belolipetskiy, V. N. Chernega, V. I. Grebenkin, and O. V. Man'ko, The quantum states of inverted and usual oscillators and particle with spin-1/2 states in probability representation of quantum mechanics, in A. Dodonov and C. C. H. Ribeiro (Eds.), Proceedings of the Second International Workshop on Quantum Nonstationary Systems, pp. 211–226 (LF Editorial, São Paulo, 2024). ISBN: 978-65-5563-446-4.