Chapter 11

Dynamics of Entropy Production and Quantum Correlations in Two-Mode Gaussian Open Systems

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We have the great pleasure to have written this paper in the honour of Professor Victor Dodonov on the occasion of his 75th Anniversary and to wish him a long life in good health and further success in his scientific activity!

11.1 Introduction

In non-equilibrium classical and quantum thermodynamics entropy production is a basic concept, closely related to the second law of thermodynamics. It enables to identify and quantify the irreversibility of physical processes, described by the generation of entropy and dissipation of heat into the environment of the systems [1–10]. The entropy change ΔS of the state of a system that exchanges energy during its interaction with a thermal bath at temperature T has a lower bound, given by the second law of thermodynamics:

$$\Delta S \geq \int \frac{\delta Q}{T}, \tag{11.1}$$

where δQ is the infinitesimal heat absorbed by the system. The strict inequality is characteristic for irreversible processes during which the energy is dissipated into the thermal reservoir in the form of heat [11]. At the same time, in addition to the entropy that flows from the system to the reservoir, some entropy, called entropy production, can be intrinsically generated in the system. According to the second law of thermodynamics the entropy production is non-negative, taking a zero value only if the system is in thermal equilibrium with its reservoir. Consequently, entropy production is used as a measure of the degree of irreversibility and to characterise the non-equilibrium phenomena. The entropy production Σ is defined as

$$\Sigma \equiv \Delta S - \int \frac{\delta Q}{T} \geq 0. \tag{11.2}$$

From Eq. (11.2) one can obtain the following relation [6, 12]:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \Pi(t) - \Phi(t), \tag{11.3}$$

where $\Pi(t)$ denotes the irreversible entropy production rate and $\Phi(t)$ the entropy flux from the system into the thermal reservoir. If the system is in a stationary state, these two quantities have strictly positive and equal values, while when both of them are zero, then it is reached the thermal equilibrium. For an open system the entropy does not satisfy a continuity equation, therefore entropy production is not a physical observable, so that, in general, it is not directly accessible.

In the last years a large interest was directed to the study of the properties of entropies of quantum states and of the thermodynamic implications of quantum features, including the understanding of the role, properties and evolution of entropy production in stochastic thermodynamics and in relation to the theory of open quantum systems [8–10, 13–16].

For a Markovian dynamics of open quantum systems, described by a quantum dynamical semigroup, the information flows monotonically from the system to the thermal reservoir and the entropy production is non-negative. There are models in which it is noticed a backflow of information from the reservoir to the system, and this behaviour is interpreted as a signature of non-Markovianity. There could be intervals of time during which the entropy production takes negative values, nevertheless it is accepted that this fact should not be considered as a violation of the second law of thermodynamics [17], but it can be interpreted as a backflow of information generated by the quantum non-Markovianity, meaning that the system retrieves some of the information that it has lost previously during its interaction with the reservoir.

In Refs. [15, 16] it was investigated the irreversibility generated in the stationary state of a system consisting of two linearly interacting quantum oscillators, each interacting with a local reservoir. It was also analyzed the behaviour of the entropy production rate by considering the non-Markovian Brownian motion in the case of an uncoupled bipartite system interacting with two independent baths, and it was established a connection between the entropy production rate and the quantum correlations present in the system. In the previous work [18] we used the formalism of the theory of open quantum systems based on completely positive dynamical semigroups [19] and investigated the dynamics of the rate of irreversible entropy production in a system consisting of two coupled non-resonant bosonic modes immersed in a common thermal environment. By extending the study performed in Refs. [15, 16], we described the behaviour of the the rate of entropy production for the initial state of the system, its evolution in time, and for the non-equilibrium stationary state of the considered system.

In the present work, by considering the same system consisting of two coupled non-resonant bosonic modes embedded in a common thermal environment, we enlarge the approach used in Ref. [18]. Specifically, since the correlations existing in a bipartite system are determined by its entropy, and therefore the dynamics of correlations is related to a production of entropy, we provide a description of these two fundamental quantum characteristics, namely we compare the behaviour of the entropy production rate with that one of Rényi-2 mutual information and of entanglement [20–24], relatively to their evolution with time and in the stationary state.

The paper is organised as follows. In Sec. 2 we write the Markovian master equation for the density operator of the open system interacting with a general environment and solve the Lyapunov evolution equation for the covariance matrix of the state of the bimodal bosonic system. In Sec. 3 we introduce the entropy production rate for Gaussian states and in Sec. 4 the Gaussian quantum correlations. In Sec. 5 we describe the dynamics of the entropy production rate and make a comparison between the behaviour of the entropy production rate and the correlations present in the considered system. Finally, we summarise the obtained results and present the conclusions in Sec. 6.

11.2 Master equation for two bosonic modes interacting with the environment

We consider a system consisting of two coupled bosonic modes (harmonic oscillators) in weak interaction with a thermal environment. In order to study its dynamics we use the formalism based on completely positive quantum dynamical semigroups, in which the Markovian irreversible time evolution of an open system is described by the Gorini-Kossakowski-Sudarshan-Lindblad master equation for the density operator $\rho(t)$ [19, 25–27]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar}\sum_j (2B_j\rho(t)B_j^\dagger - \{\rho(t), B_j^\dagger B_j\}_+), \tag{11.4}$$

where H is the Hamiltonian of the open system and the operators $B_j, B_j^\dagger$, which are defined on the Hilbert space of H, describe the interaction of the system with the environment.

The Hamiltonian of two linearly coupled in coordinates bosonic modes of frequencies ω_1 and ω_2 is

$$H = \frac{\hbar\omega_1}{2}(x^2 + p_x^2) + \frac{\hbar\omega_2}{2}(y^2 + p_y^2) + qxy, \tag{11.5}$$

where x, y, p_x, p_y are the dimensionless position and, respectively, momentum operators of the two modes and q is the coupling. The operators B_j are taken as polynomials of first degree in these canonical operators, and by choosing initial Gaussian states, then the linear character of the dynamics assures the preservation of the Gaussianity in time [28, 29]. We denote by $\mathbf{R} = \{x, p_x, y, p_y\}^{\mathrm{T}}$ the vector of canonically conjugated quadrature operators of the two bosonic modes and by σ the 4×4 bimodal covariance matrix, whose elements are given by the second statistical moments of the quadrature operators:

$$\sigma_{ij} = \mathrm{Tr}[(R_iR_j + R_jR_i)\rho], i, j = 1, \ldots, 4, \tag{11.6}$$

which fully characterise any Gaussian state of a bimodal system. We neglect here the first moments, since they can be made zero by performing suitable local displacements in the phase space.

The time evolution of the covariance matrix $\sigma(t)$ is described by the following Lyapunov equation [27]:

$$\frac{d\sigma(t)}{dt} = A\sigma(t) + \sigma(t)A^{\mathrm{T}} + D, \tag{11.7}$$

$$A = \begin{pmatrix} -\lambda & \omega_1 & 0 & 0 \\ -\omega_1 & -\lambda & -q & 0 \\ 0 & 0 & -\lambda & \omega_2 \\ -q & 0 & -\omega_2 & -\lambda \end{pmatrix}, \tag{11.8}$$

where A is the drift matrix, D is the diffusion matrix and λ is the dissipation parameter. The diffusion matrix takes the following form [19, 27] (we set $\hbar = 1$):

$$D = 2\ \mathrm{diag}\{\lambda \coth\frac{\omega_1}{2k_BT}, \lambda \coth\frac{\omega_1}{2k_BT}, \lambda \coth\frac{\omega_2}{2k_BT}, \lambda \coth\frac{\omega_2}{2k_BT}\}, \tag{11.9}$$

where k_B is the Boltzmann constant and T is the temperature of the thermal bath.

The time-dependent solution of Eq. (11.7) is [27]

$$\sigma(t) = M(t)[\sigma(0) - \sigma_{\mathrm{s}}]M^{\mathrm{T}}(t) + \sigma_{\mathrm{s}}, \tag{11.10}$$

where $M(t) \equiv \exp(At)$.

The evolution due to the Gaussian completely positive map is determined by the two 4 x 4 matrices M and $Y = \sigma_{\mathrm{s}} - M\sigma_{\mathrm{s}}M^{\mathrm{T}}$, which satisfy $Y + i\Omega \geq iM\Omega M^{\mathrm{T}}$, where Ω is the symplectic matrix $\Omega = \bigoplus_1^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The unitary evolution generated by the Hamiltonian (11.5) does not commute with the dynamics generated by the interaction of the system with the thermal bath, therefore the coupling between the two bosonic modes influences the irreversibility. The considered open system evolves in the limit of large times to a non-equilibrium steady state, determined by the stationary covariance matrix σ_{s} that can be obtained by putting $\frac{d\sigma(t)}{dt} = 0$ in Eq. (11.7):

$$A\sigma_{\mathrm{s}} + \sigma_{\mathrm{s}}A^{\mathrm{T}} = -D. \tag{11.11}$$

The matrices A and D can be considered the drift and diffusion matrices of a quantum system if and only if $D + iA\Omega^{\mathrm{T}} - i\Omega A^{\mathrm{T}} \geq 0$, which is derived using the uncertainty principle [30]. If the condition $A + A^{\mathrm{T}} < 0$ is satisfied, then the system is stable, i.e. it admits a steady state.

The considered model is analytically solvable and the covariance matrix $\sigma(t)$ depends on the chosen initial state, the parameters of the system and the environment, and on the coupling between the two modes. In the case of non-coupled bosonic modes ($q = 0$), the diffusion matrix leads to an asymptotic Gibbs state describing a thermal equilibrium with the reservoir [19, 24]. However, if the modes are coupled, then the stationary state is not anymore a product state, and in the simple case of resonant ($\omega_1 = \omega_2 = \omega$) modes its covariance matrix is given by (we set $k_B = 1$)

$$\sigma_{\rm s}(\infty) = \frac{\coth\left(\frac{\omega}{2T}\right)}{2\left(L^2 - q^2\omega^2\right)} \times \begin{pmatrix} 2L^2 - q^2\omega^2 & \lambda q^2\omega & -q\omega L & -\lambda qL \\ \lambda q^2\omega & 2L^2 + q^2(\lambda^2 - 2\omega^2) & -\lambda qL & q\omega(L - q^2) \\ -q\omega L & -\lambda qL & 2L^2 - q^2\omega^2 & \lambda q^2\omega \\ -\lambda qL & q\omega(L - q^2) & \lambda q^2\omega & 2L^2 + q^2(\lambda^2 - 2\omega^2) \end{pmatrix}, \tag{11.12}$$

where $L \equiv \omega^2 + \lambda^2$. If $q = 0$, then Eq. (11.12) becomes $\sigma_{\rm G}(\infty) = \coth\left(\frac{\omega}{2T}\right) I$.

11.3 Entropy production for Gaussian states

Since the dynamics of open quantum systems determined by the master equation (11.4) can be expressed in terms of the Fokker-Plank equation for the Wigner distribution function, it is reasonable to describe the behaviour of the entropy production by employing a corresponding approach, based on the phase space formalism [6, 31, 32]. Therefore, we define the Wigner entropy production rate [12] by

$$\Pi(t) \equiv -\partial_t K(W(t)||W_{\rm s}), \tag{11.13}$$

where $K(W(t)||W_{\rm s})$ denotes the Wigner relative entropy, $W(t)$ is the time-dependent Wigner function and $W_{\rm s}$ is the Wigner function of the stationary state.

By introducing the symplectic matrix representing the time reversal operator $E = \mathrm{diag}(1, -1, 1, -1)$, we split the dynamical variables according to their time symmetry. Correspondingly, the drift matrix A (11.8) is divided into the irreversible component $A^{\rm irr}$, given by $A^{\rm irr} = \frac{1}{2}\left(A + EAE^{\rm T}\right)$, and the reversible component $A^{\rm rev} = \frac{1}{2}\left(A - EAE^{\rm T}\right)$:

$$A^{\rm irr} = \mathrm{diag}\left(-\lambda, -\lambda, -\lambda, -\lambda\right), \tag{11.14}$$

$$A^{\rm rev} = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & -q & 0 \\ 0 & 0 & 0 & \omega_2 \\ -q & 0 & -\omega_2 & 0 \end{pmatrix}. \tag{11.15}$$

Then we obtain the following expression for the entropy production rate $\Pi(t)$ [6, 15, 16]:

$$\Pi(t) = \frac{1}{2}\mathrm{Tr}[\sigma^{-1}(t)D] + 2\mathrm{Tr}[A^{\rm irr}] + 2\mathrm{Tr}[(A^{\rm irr})^{\rm T}D^{-1}A^{\rm irr}\sigma(t)]. \tag{11.16}$$

In particular, if the system is in the non-equilibrium stationary state $\sigma_{\rm s}$, then the expression (11.16) becomes [15]

$$\Pi_{\rm s} = \mathrm{Tr}[A^{\rm irr}] + 2\mathrm{Tr}[(A^{\rm irr})^{\rm T}D^{-1}A^{\rm irr}\sigma_{\rm s}]. \tag{11.17}$$

11.4 Gaussian quantum correlations

For a two-mode continuous variable system, a Gaussian state ρ is described by the positive Wigner distribution in phase space [33]:

$$W_\rho(\boldsymbol{R}) = \frac{1}{\pi^2\sqrt{\det\sigma}} \exp\left(-\boldsymbol{R}^\intercal \sigma^{-1} \boldsymbol{R}\right), \tag{11.18}$$

where $\boldsymbol{R} \in \mathbb{R}^4$ and σ is the covariance matrix of the Gaussian state. The standard way to quantify entropy is by using the von Neumann entropy. However, one can obtain an alternative quantifier of the quantum information contained in a Gaussian state, by employing the Shannon entropy of the Wigner function (11.18):

$$\mathcal{S}_\sigma(\rho) = \frac{1}{2}\ln(\det\ \sigma) + 2(1 + \ln\pi). \tag{11.19}$$

In quantum information theory it was introduced the following family of additive entropies, called Rényi-α entropies:

$$\mathcal{S}_\alpha(\rho) = (1-\alpha)^{-1}\ln(\mathrm{Tr}\,\rho^\alpha),\ \ \alpha \geq 0, \tag{11.20}$$

which are widely used to study the quantum correlations in multipartite systems. For Gaussian states the expression in (11.19) coincides, up to an additive constant, with the Rényi entropy of order 2, given by Eq. (11.20) for $\alpha = 2$. In the limit $\alpha = 1$ the expression in (11.20) becomes the von Neumann entropy $\mathcal{S}_1(\rho) = -\mathrm{Tr}(\rho\ln\rho)$, while for $\alpha = 2$ the Rényi entropy becomes $\mathcal{S}_2(\rho) = -\ln(\mathrm{Tr}\ \rho^2)$, that is related to the purity of the state ρ. For Gaussian states, using Eq. (11.18), the Rényi-2 entropy becomes [34]:

$$\mathcal{S}_2(\rho) = \frac{1}{2}\ln(\det\sigma), \tag{11.21}$$

which gives 0 for pure states ($\det\ \sigma = 1$) and increases with the mixedness of the state. By comparing this expression with Eq. (11.19), we indeed notice that Rényi and Shannon entropy coincide, up to an additive constant. Rényi-2 entropy has all the properties required for a legitimate measure of entropy, including the strong subadditivity [34].

11.4.1 Rényi-2 mutual information

For a bipartite Gaussian state ρ of a system composed of subsystems A and B, the Gaussian Rényi-2 mutual information is defined as

$$\mathcal{I}\left(\rho_{A:B}\right) = \mathcal{S}_2\left(\rho_A\right) + \mathcal{S}_2\left(\rho_B\right) - \mathcal{S}_2\left(\rho\right), \tag{11.22}$$

where ρ_A and ρ_B are the two marginals of ρ. By writing the corresponding two-mode covariance matrix σ in block form, in terms of the covariance matrices of the subsystems:

$$\sigma = \begin{pmatrix} \sigma_A & \sigma_C \\ \sigma_C^{\mathrm{T}} & \sigma_B \end{pmatrix}, \tag{11.23}$$

the expression of the Gaussian Rényi-2 mutual information becomes [34]

$$\mathcal{I}\left(\rho_{A:B}\right) = \frac{1}{2}\ln\left(\frac{\det\sigma_A\det\sigma_B}{\det\sigma}\right). \tag{11.24}$$

$\mathcal{I}\left(\rho_{A:B}\right)$ takes positive values and it is a measure of the total correlations in the bipartite state ρ.

11.4.2 Quantum entanglement

As a measure of the entanglement we use the logarithmic negativity, given by

$$\mathcal{L} = \max\{0, -\ln \tilde{\nu}_-\}, \tag{11.25}$$

where $\tilde{\nu}_-$ is the smallest symplectic eigenvalue of the partially transposed covariance matrix, which contains all the required information to quantify the entanglement of two-mode Gaussian states. It is given by the following expression, in terms of the covariance matrices [30]:

$$2\tilde{\nu}_-^2 = \det \sigma_A + \det \sigma_B - 2\det \sigma_C - \sqrt{(\det \sigma_A + \det \sigma_B - 2\det \sigma_C)^2 - 4\det \sigma}. \tag{11.26}$$

11.5 Dynamics of the entropy production rate and quantum correlations

As stated previously, the degree of irreversibility of the dynamics of an open system can be evaluated by the entropy production rate of its state. In the following we will describe the dynamics of the entropy production rate, as a function of the coupling between the two bosonic modes and the parameters characterising the initial Gaussian state of the considered system and the thermal reservoir. Likewise, we will describe the evolution with time of the measures of the mentioned correlations between the two modes, namely Gaussian Rényi-2 mutual information and logarithmic negativity, as well as their behaviour in the stationary state, and we will make a comparison with the behaviour of the entropy production rate.

We consider an initial squeezed thermal state with the covariance matrix given by

$$\sigma_0 = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{pmatrix}, \tag{11.27}$$

where

$$\begin{aligned} a &= 2n_1 \cosh^2 r + 2n_2 \sinh^2 r + \cosh 2r, \\ b &= 2n_1 \sinh^2 r + 2n_2 \cosh^2 r + \cosh 2r, \\ c &= (n_1 + n_2 + 1)\sinh 2r, \end{aligned} \tag{11.28}$$

n_1 and n_2 are the average thermal photon numbers of the modes and r is the squeezing parameter of the initial state.

11.5.1 Time evolution of the entropy production rate and Gaussian quantum correlations

In this Subsection we describe the behaviour of the entropy production rate as a function of time, coupling between the two bosonic modes and the parameters characterising the initial state of the system and the thermal reservoir. Its general analytical expression is cumbersome, therefore, we report here only the expression of the entropy production rate in the case of uncoupled bosonic modes ($q = 0$), for an initial squeezed vacuum state, in the resonant case ($\omega_1 = \omega_2 \equiv \omega$) [18]:

$$
\begin{aligned}
\Pi(t) \quad = \quad & 4\lambda[-1 + \mathrm{e}^{-2t\lambda}\left(-1 + \cosh 2r \tanh \frac{\omega}{2T}\right) \\
& + \left(\mathrm{e}^{2t\lambda}((-1 + \mathrm{e}^{2t\lambda})(1 + \cosh \frac{\omega}{T}) + \cosh 2r \sinh \frac{\omega}{T})\right) / \\
& \left(\mathrm{e}^{2t\lambda}(-2 + \mathrm{e}^{2t\lambda}) + (2 - 2\mathrm{e}^{2t\lambda} + \mathrm{e}^{4t\lambda}) \cosh \frac{\omega}{T} + 2(-1 + \mathrm{e}^{2t\lambda}) \cosh 2r \sinh \frac{\omega}{T}\right)].
\end{aligned}
\tag{11.29}
$$

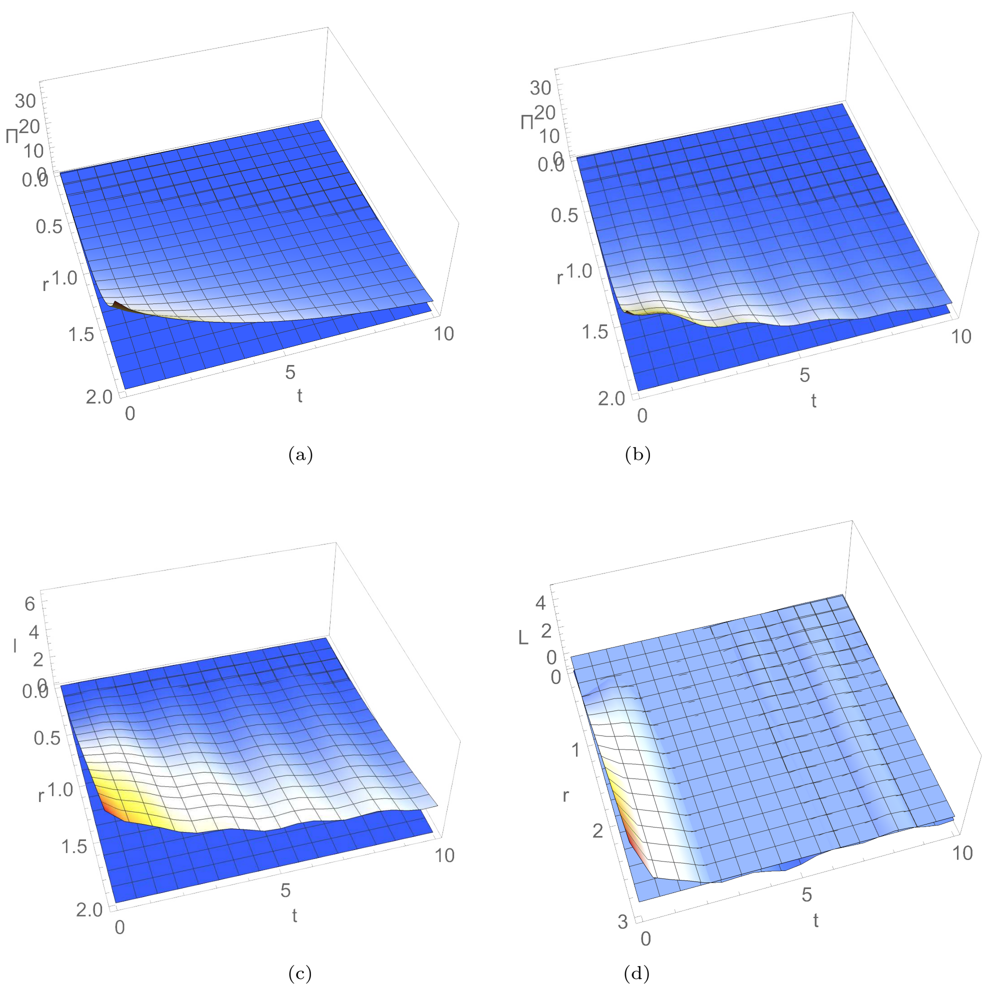


Figure 11.1: The dependence on time t and squeezing r for non-resonant modes with frequencies $\omega_1 = 1$, $\omega_2 = 1.7$ of: entropy production rate for dissipation $\lambda = 0.1$, uncoupled modes with $q = 0$ (a) and coupled modes with $q = 0.2$ (b); mutual information for $q = 0.2$ and dissipation $\lambda = 0.1$ (c), and logarithmic negativity for $q = 0.4$ and dissipation $\lambda = 0.2$ (d). The initial state is a symmetric squeezed thermal state with average thermal photon number $n = 1$ and the temperature of the thermal bath is $T = 0.1$.

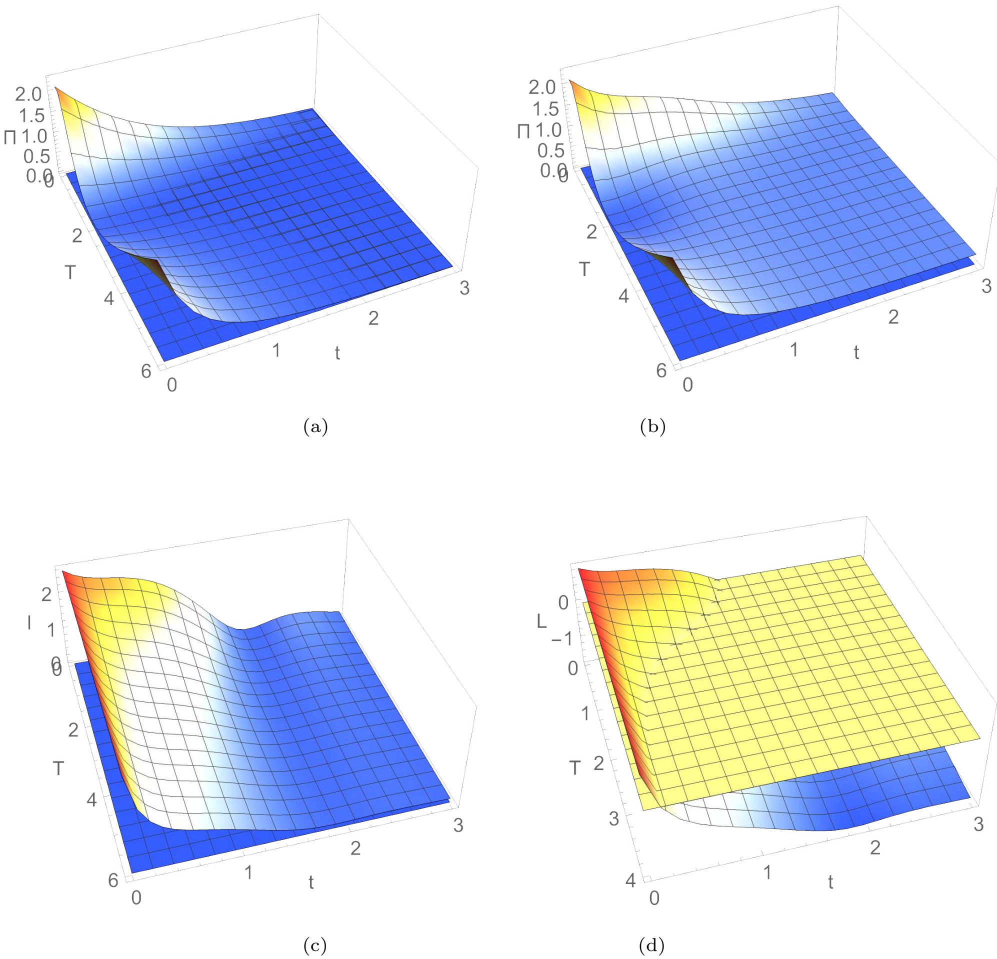


Figure 11.2: The dependence on time t and temperature T for non-resonant modes with frequencies $\omega_1 = 1, \omega_2 = 1.7$ of: entropy production rate for uncoupled modes with $q = 0$ (a) and coupled modes with $q = 0.8$ (b) for an initial symmetric thermal state (squeezing $r = 0$) and average thermal photon number $n = 1$; mutual information for $q = 0.8$ (c), and logarithmic negativity for $q = 0.8$ (d) for an initial symmetric squeezed thermal state with squeezing $r = 1$ and average thermal photon number $n = 1$. The dissipation parameter is $\lambda = 0.4$.

The evolution with time of the entropy production rate $\Pi(t)$ is illustrated in Fig. 11.1 for an initial symmetric squeezed thermal state (11.27), (11.28) as a function of the squeezing of the initial state, in both cases of (a) uncoupled and (b) coupled non-resonant bosonic modes. We notice that $\Pi(t)$ is always positive in the considered Markovian approximation and it is decreasing in time. At a given moment of time, $\Pi(t)$ is increasing

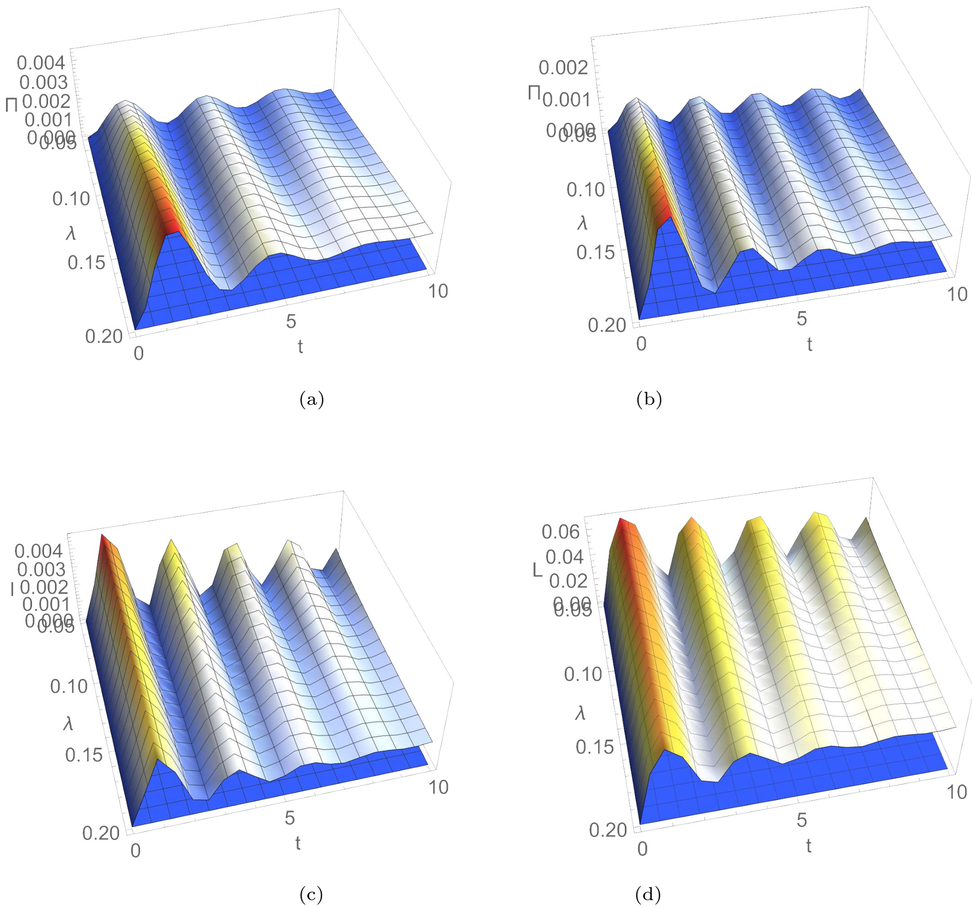


Figure 11.3: The dependence on time t and dissipation λ for an initial coherent state of: entropy production rate for resonant modes with frequency $\omega = 1$ (a) and non-resonant modes with frequencies $\omega_1 = 1, \omega_2 = 1.7$ (b); mutual information (c), and logarithmic negativity (d), both for non-resonant modes with frequencies $\omega_1 = 1, \omega_2 = 1.7$. The coupling constant is $q = 0.1$ and the temperature is $T = 0.1$.

with the squeezing of the initial state.

In Fig. 11.1 (c) and (d) the evolution with time of the mutual information $\mathcal{I}(t)$ and logarithmic negativity $\mathcal{L}(t)$ is illustrated, respectively, as a function of the squeezing of the initial state. We observe that they are also decreasing in time. At a given moment of time, $\mathcal{I}(t)$ and $\mathcal{L}(t)$ are increasing with the squeezing of the initial state, like the entropy production rate $\Pi(t)$. In addition, entanglement manifests the sudden death phenomenon, that means that its survival time is finite in the case of uncoupled modes and non-zero temperature. However, if the modes are coupled, then the entanglement can survive in the asymptotic stationary state for definite values of the coupling between the modes, of their

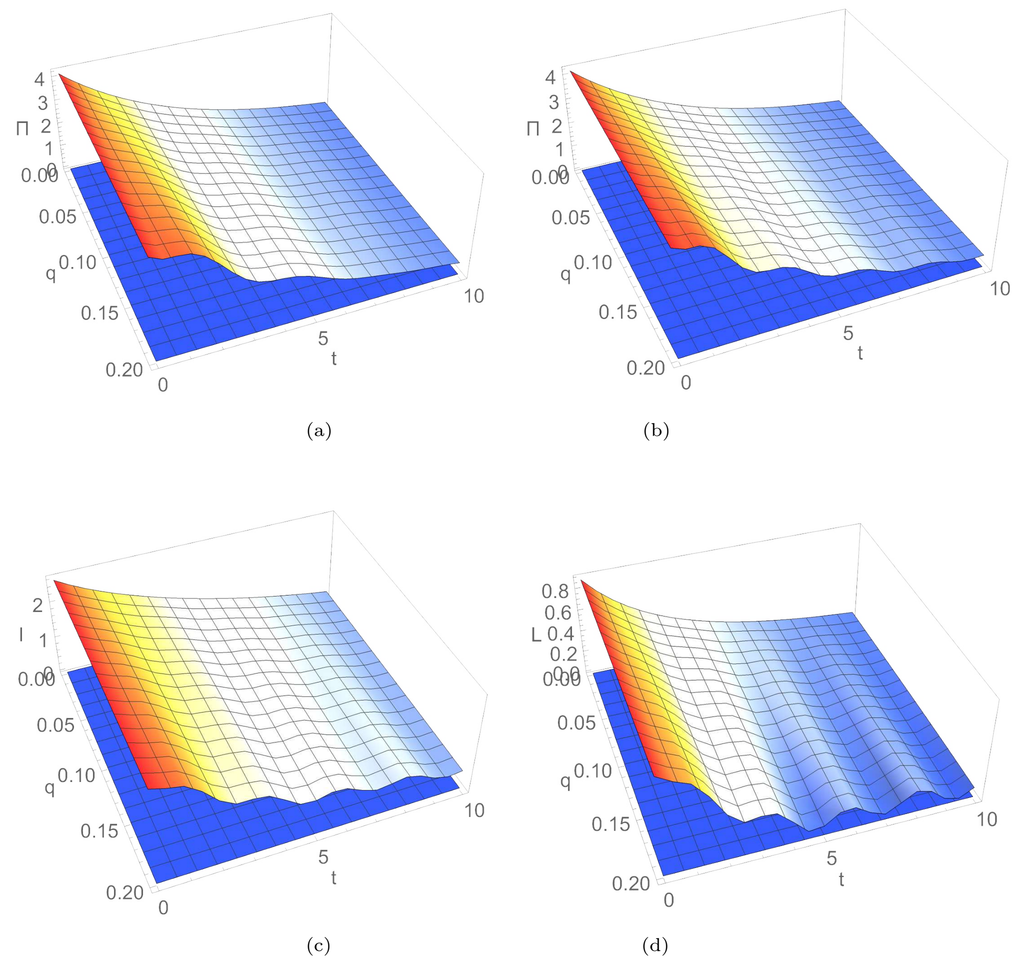
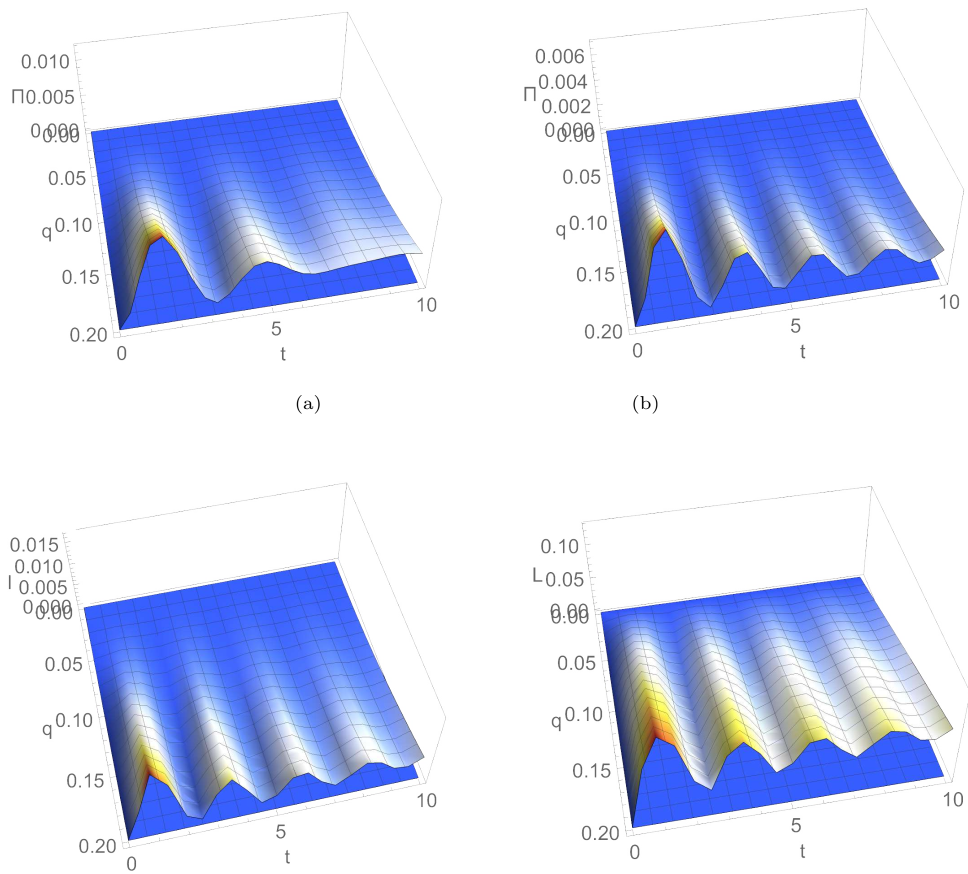


Figure 11.4: The dependence on time t and coupling q between the modes of: entropy production rate for resonant modes with frequency $\omega = 1$ (a) and non-resonant modes with frequencies $\omega_1 = 1, \omega_2 = 1.7$ (b); mutual information (c), and logarithmic negativity (d), both for non-resonant modes with frequencies $\omega_1 = 1, \omega_2 = 1.7$. The initial state is a symmetric squeezed thermal state with squeezing $r = 1$ and average thermal photon number $n = 1$. The parameters of the thermal bath are temperature $T = 0.1$ and dissipation $\lambda = 0.1$.

frequencies, and of the parameters characterising the thermal reservoir – temperature and dissipation.

The evolution with time of the entropy production rate $\Pi(t)$ is illustrated in Fig. 11.2 for an initial symmetric thermal state as a function of the temperature of the reservoir, in both cases of (a) uncoupled and (b) coupled non-resonant bosonic modes. We observe that $\Pi(t)$ is decreasing by increasing the temperature of the thermal reservoir for relatively small values of the temperature, while for larger values it is increasing with the tempera-



(a) (b) (c) (d)

Figure 11.5: The dependence on time t and coupling q between the modes for an initial coherent state of: entropy production rate for resonant modes with frequency $\omega = 1$ (a) and non-resonant modes with frequencies $\omega_1 = 1, \omega_2 = 1.7$ (b); mutual information (c), and logarithmic negativity (d), both for non-resonant modes with frequencies $\omega_1 = 1, \omega_2 = 1.7$. The parameters of the thermal bath are temperature $T = 0.1$ and dissipation $\lambda = 0.1$.

ture. This behaviour, at a given moment of time, is the result of the competition between the influences provided by the thermal photon number of the modes and the temperature of the thermal bath. In Fig. 11.2 (c) and (d) it is illustrated the evolution with time of the mutual information $\mathcal{I}(t)$ and logarithmic negativity $\mathcal{L}(t)$, respectively, as a function of the temperature of the reservoir. We see that they are decreasing by increasing the temperature of the thermal reservoir. For relatively large values of the temperature one can notice that the entropy production rate $\Pi(t)$ has a similar behaviour.

In Fig. 11.3 we illustrate the evolution with time of the entropy production rate $\Pi(t)$ as a function of the dissipation parameter, for an initial coherent state, in the case of both

(a) resonant and (b) non-resonant modes. One can notice that $\Pi(t)$ is increasing with the dissipation rate of the environment, and this behaviour can be interpreted as a signature of the increase of the degree of irreversibility with the losses generated during the interaction of the open system with its thermal environment. In Fig. 11.3 (c) and (d) it is presented the evolution with time of the mutual information $\mathcal{I}(t)$ and logarithmic negativity $\mathcal{L}(t)$, respectively, as a function of the dissipation rate, for an initial coherent state. We observe that, in contrast to the entropy production rate $\Pi(t)$, they are decreasing by increasing the dissipation rate of the bath. In addition, one can see that $\mathcal{I}(t)$ and $\mathcal{L}(t)$ are zero at the initial moment of time, and after this moment they have non-zero values, that is we are witnessing their generation, due to the coupling between the bosonic modes.

The evolution with time of the entropy production rate $\Pi(t)$ as a function of the coupling between the two bosonic modes, in both cases of resonant and non-resonant modes, is illustrated in Fig. 11.4 (a), (b) for an initial symmetric squeezed thermal state and in Fig. 11.5 (a), (b) for an initial coherent state. For the same initial states, the evolution with time of mutual information $\mathcal{I}(t)$ and logarithmic negativity $\mathcal{L}(t)$ as a function of the coupling between the modes is shown in Figs. 11.4 (c), 11.5 (c), and Figs. 11.4 (d), 11.5 (d), respectively. We notice that both these correlations are increasing with the coupling q between the modes, as expected, like the behaviour of $\Pi(t)$.

The values of the parameters used in the presented plots are chosen in agreement with the Markovian condition of weak coupling λ of the system to the thermal environment and with the already mentioned condition of stability $A + A^{\mathrm{T}} < 0$. With these assumptions we observe that $\Pi(t)$ is increasing with the coupling q between the modes. Consequently, the stronger the coupling between the modes, the more irreversible is the corresponding stationary process. It follows that the coupling between the modes plays a crucial role relatively to the entropy production rate in the stationary state. Indeed, for zero coupling between the modes, the entropy production rate is zero in the stationary state, when the system is actually in thermal equilibrium with the environment. By contrary, for non-zero coupling between the modes, $\Pi(t)$ tends asymptotically in time to a non-zero value in the non-equilibrium stationary state. From the figures presented in this Subsection it results that, depending on the chosen parameters, the evolution with time is monotonous and can also manifest oscillations, which are relatively more dense and more intense for non-resonant modes compared to the resonant case. We observe also that the state of non-resonant modes has, in general, a larger rate of entropy production, compared to the resonant case. This behaviour could be interpreted as a result of the breaking of the symmetry between the two subsystems, which is considered a factor leading to the increase of the entropy production rate. Likewise, squeezing, which represents a useful resource in both quantum information and quantum thermodynamics, is intimately related to the Heisenberg uncertainty principle and, by introducing an asymmetry between the position and momentum uncertainties, it modifies the energy fluctuations and introduces an extra increase in entropy and, therefore, the entropy production is also modified [35–39]. The presented behaviour is the result of the competition between the influences exerted by the parameters characterising the initial state (thermal photon number and frequency of the modes) and the thermal reservoir (temperature and dissipation) on the entropy production rate and quantum correlations.

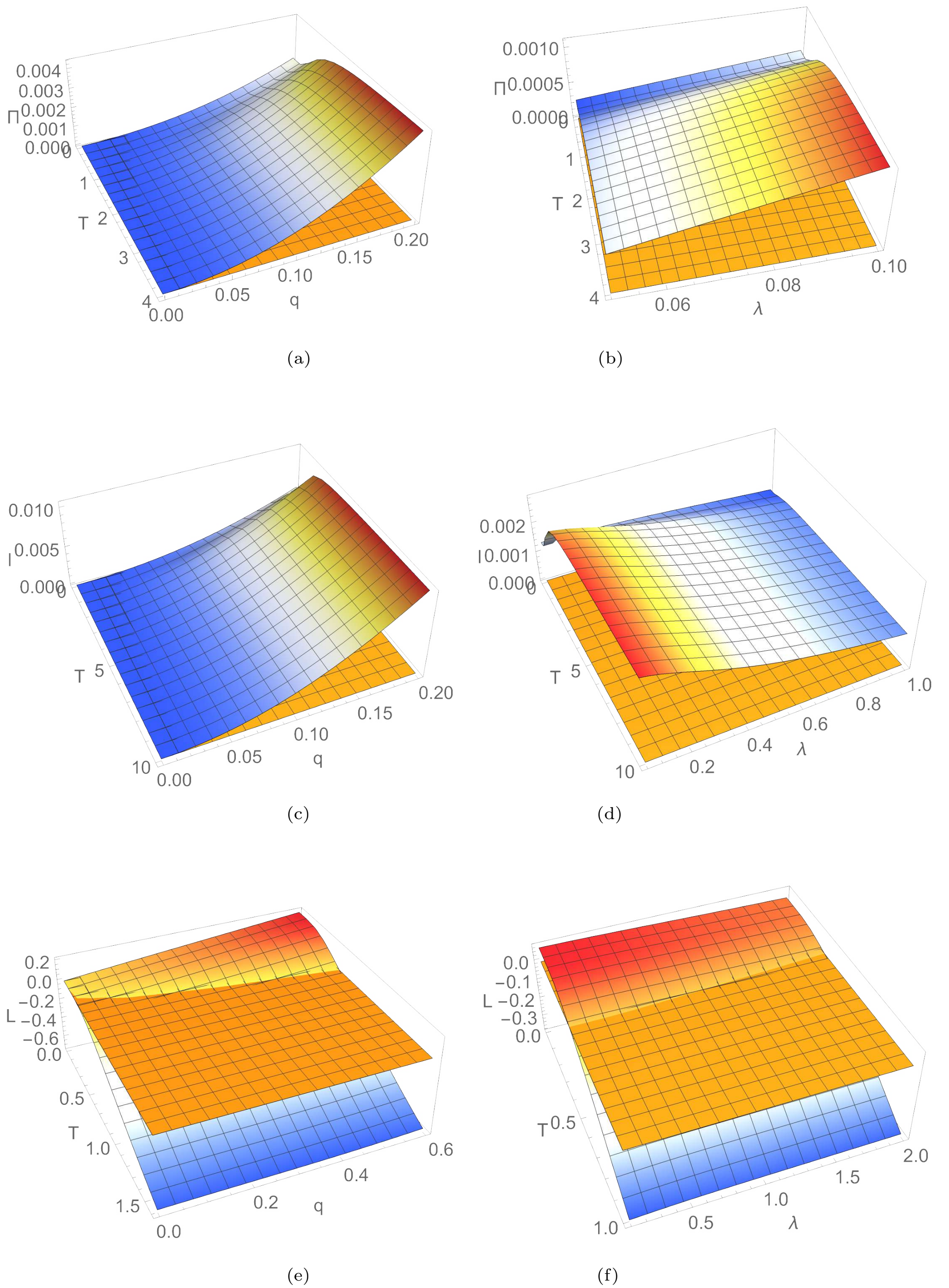


Figure 11.6: Dependence on temperature T of the reservoir and coupling q between the modes (left) and dissipation λ (right) in the stationary state of: entropy production rate, for $\lambda = 0.1$ (a) and $q = 0.1$ (b); mutual information, for $\lambda = 0.1$ (c) and $q = 0.1$ (d); logarithmic negativity, for $\lambda = 0.3$ (e) and $q = 0.2$ (f). The frequencies of the modes are $\omega_1 = 1, \omega_2 = 1.7$.

11.5.2 Entropy production rate and Gaussian quantum correlations in the stationary state

The expression of the entropy production rate in the stationary state has the following form [18]:

$$\Pi_s = \big[\lambda q^2 (q^2 \left(\omega_1^4 + 18\omega_1^2\omega_2^2 + \omega_2^4 + 10\lambda^2(\omega_1^2+\omega_2^2)\right) - 16\omega_1\omega_2(\lambda^2+\omega_1^2)(\lambda^2+\omega_2^2)$$
$$+(16\lambda^6 + 20\lambda^4(\omega_1^2+\omega_2^2) + \omega_1\omega_2(-3q^2+4\omega_1\omega_2)(\omega_1^2+\omega_2^2)$$
$$+\lambda^2(4\omega_1^4 - 6q^2\omega_1\omega_2 + 24\omega_1^2\omega_2^2 + 4\omega_2^4))$$
$$\times\left((\coth\frac{\omega_1}{2T})^2 + (\coth\frac{\omega_2}{2T})^2\right)\tanh\frac{\omega_1}{2T}\tanh\frac{\omega_2}{2T})\big]$$
$$\times\big[2((4\lambda^2+\omega_1^2)^2 + 4q^2\omega_1\omega_2 + 2(4\lambda^2-\omega_1^2)\omega_2^2 + \omega_2^4)$$
$$\times(\lambda^4 + \omega_1\omega_2(\omega_1\omega_2 - q^2) + \lambda^2(\omega_1^2+\omega_2^2))\big]^{-1}. \quad (11.30)$$

In the resonant case it becomes:

$$\Pi_{sr} = \frac{q^2\lambda(\lambda^2+\omega^2)}{(\lambda^2+\omega^2)^2 - q^2\omega^2}. \quad (11.31)$$

For the stationary state, in the same case of resonant modes, the mutual information has the following expression, which does not depend on temperature:

$$\mathcal{I}_s = \frac{1}{2}\ln\frac{(\lambda^2+\omega(\omega-q))(\lambda^2+\omega(\omega+q))(q^4+8q^2(\lambda^2-\omega^2)+16(\lambda^2+\omega^2)^2)}{4(q^2(\lambda^2-\omega^2)+2(\lambda^2+\omega^2)^2)^2}. \quad (11.32)$$

We see that the entropy production rate in the stationary state does not depend on the initial state, as expected. In addition, we notice that in the resonant case Π_{sr} does not depend also on the temperature of the thermal bath. The open system consisting of two resonant bosonic modes has a symmetric configuration, therefore the entropy production rate of the non-equilibrium stationary state to which it evolves has the minimum possible value, and this non-equilibrium state is the closest possible to an equilibrium state [15].

In Fig. 11.6 (a), (b) we illustrate the dependence of Π_s on the coupling between the bosonic modes and on the parameters characterising the thermal environment – dissipation and temperature. We can see that the entropy production rate in the stationary state is increasing with the coupling between the modes and with the dissipation. We observe also that in the non-resonant case it slightly increases with the temperature of the reservoir for relatively small values, and it saturates for larger values of temperature. If the two modes are uncoupled, then it follows from Eq. (11.30) that the entropy production rate in the stationary state is zero and the system relaxes from the non-equilibrium stationary state to the equilibrium Gibbs thermal state, in agreement with the previously obtained results [13, 16, 40, 41].

The coupling between the modes plays a crucial role relatively not only to the entropy production rate Π_s in the stationary state. Indeed, we have seen that, for zero coupling between the modes, $\Pi_s = 0$ in the stationary state, when the system is actually in thermal equilibrium with the bath. The same is true for both mutual information, which tends asymptotically to zero in the limit of large times, and also for the quantum entanglement, which suddenly vanishes in a finite time during the interaction of the system with the reservoir, in the case of uncoupled modes and for non-zero temperature of the bath. By contrary, for coupled modes entropy production rate and mutual information tend asymptotically in time to a non-zero value in the non-equilibrium stationary state, while entanglement survives in the stationary state only for definite values of the coupling between the modes, temperature and dissipation rate, determined by the competition between the influences produced by these parameters.

From Fig. 11.6 (c), (d) we notice that in the non-resonant case the mutual information in the stationary state, like the entropy production rate, slightly increases with the temperature for relatively small values, and it saturates for larger values of temperature. Concerning the entanglement, from Fig. 11.6 (e), (f) we can see that the logarithmic negativity is decreasing by increasing the temperature. For a given value of the coupling between the modes, entanglement survives in the stationary state only for values of temperature smaller than a definite value, while for larger values of the temperature the entanglement is completely lost. Like the entropy production rate, mutual information and the logarithmic negativity in the stationary state are increasing with the coupling between the modes, as illustrated in Fig. 11.6 (c), (e). We can say that the more correlations are shared between the two bosonic modes, the larger the irreversibility, since the larger is the entropy generated in the stationary state. In particular, from Fig. 11.6 (a), (c), (e) we see that in the case of two uncoupled bosonic modes, when they separately reach local thermal equilibrium states, the values of the entropy production rate and of both considered correlations vanish. From Fig. 11.6 (d), (f) we notice that, differently form the behaviour of the entropy production rate illustrated in Fig. 11.6 (b), mutual information and the logarithmic negativity are decreasing by increasing the dissipation, as expected, due to the destroying effect of the environment.

These results confirm the ones obtained in Refs. [15, 16, 18] concerning the relation between the entropy production rate and the correlations existing between the two bosonic modes of the considered system.

11.6 Conclusions

We have investigated the Markovian time evolution of the entropy production rate as an indicator of the irreversibility generated in a bipartite quantum system composed of two coupled bosonic modes immersed in a thermal reservoir. We have studied the dynamics of the system in the framework of the theory of open quantum systems based on completely positive dynamical semigroups, for initial two-mode squeezed thermal states and coherent states. We also described the evolution with time and the behaviour in the stationary state of the correlations shared between the two modes, namely Gaussian Rényi-2 mutual information and entanglement, and made a comparison with the behaviour of the entropy production rate.

We have shown that the behaviour and evolution of the rate of the entropy production and of the mentioned correlations strongly depend on the squeezing parameter of the initial Gaussian state, frequencies of the modes, the parameters characterising the thermal reservoir (temperature and dissipation rate) and the strength of the coupling between the two modes.

In the case of the Markovian dynamics of open quantum systems the information flows from the system into the environment and, correspondingly, the rate of entropy production is a positive quantity. We have shown that the entropy production rate is always increasing with the squeezing between the modes and with the dissipation rate; its evolution with time is monotonous and may also present oscillations, which are relatively more dense and more intense in the case of non-resonant modes, compared to the resonant case. Squeezing introduces an asymmetry between the position and momentum uncertainties of the modes, that modifies the energy fluctuations and introduces an extra increase in entropy, and this leads, consequently, to the increase of the rate of entropy production. The increase of the entropy production rate with the dissipation can be interpreted as a signature of the increase of the degree of irreversibility with the losses generated during the interaction of the subsystem with its thermal environment. In comparison, mutual information and logarithmic negativity are increasing with the squeezing of the initial state, like the entropy production rate, while they are decreasing by increasing the dissipation rate, in contrast to the entropy production rate. In addition, entanglement manifests the sudden death phenomenon, so that its survival time is finite in the case of uncoupled modes and non-zero temperature.

The entropy production rate, mutual information and logarithmic negativity are increasing with the coupling between the modes. Consequently, the stronger the coupling between the modes, and therefore the stronger the correlations between the modes, the more irreversible is the corresponding evolution and the stationary process, that is the larger the entropy generated in the system during its interaction with the environment. The coupling is crucial relatively to all these quantities in the stationary state: if the coupling between the modes tends to zero, then the entropy production rate tends to zero in the stationary state, and the system relaxes from a non-equilibrium stationary state toward the equilibrium Gibbs thermal state. The same is true for mutual information which tends asymptotically to zero in the limit of large times, for uncoupled modes. Concerning the entanglement, it suddenly disappears in a finite time during the interaction of the system with reservoir, in the case of uncoupled modes and for non-zero temperature of the reservoir. By contrary, for non-zero coupling between the modes, entropy production rate and mutual information tend asymptotically in time to a non-zero value in the non-equilibrium stationary state, while entanglement survives in the asymptotic stationary state only for definite values of the coupling between the modes, temperature and dissipation rate, determined by the competition between the influences produced by these parameters.

The entropy production rate in the stationary state is increasing with the coupling between the modes, with the dissipation, and slightly with the temperature of the reservoir for relatively small values, and it saturates for larger values of temperature (in the non-resonant case), but it does not depend on the initial state. In the non-resonant case the mutual information in the stationary state, like the entropy production rate, slightly increases with the temperature for relatively small values, and it saturates for larger values of temperature. Concerning the entanglement, if it survives in the stationary state, then the logarithmic negativity is decreasing by increasing the temperature. We remind that for a given value of the coupling between the modes, entanglement survives in the stationary state only for values of the temperature smaller than a definite value, while for larger values of the temperature the entanglement is completely lost.

In the stationary state mutual information and the logarithmic negativity are increasing with the coupling between the modes, like the entropy production rate, while, differently form the behaviour of the entropy production rate, they are decreasing by increasing the dissipation, as expected, due to the destroying effect of the thermal reservoir.

The obtained results confirm and at the same time are complementary to those ones presented in Refs. [15, 16, 18], and they emphasise the closed relation between the irreversibility, which quantifies the departure from quasi-static reversible transformations generated by the dynamical and stationary process, and the correlations existing in the considered bipartite system.

Acknowledgments

The authors acknowledge the financial support received from the Romanian Ministry of Research, Innovation and Digitisation, through the Project PN 23 21 01 01/2023.

11.7 References

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Cite this work as:

T. Mihaescu and A. Isar, Dynamics of Entropy Production and Quantum Correlations in Two-Mode Gaussian Open Systems, in A. Dodonov and C. C. H. Ribeiro (Eds.), Proceedings of the Second International Workshop on Quantum Nonstationary Systems, pp. 175–194 (LF Editorial, São Paulo, 2024). ISBN: 978-65-5563-446-4.