Chapter 9

Quantum regularisations of metric tensors

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9.1 Introduction

The content of this contribution draws heavily from recent articles, specifically [1] and [2]. These sources provide the foundational framework and methodologies for the non-orthodox quantization of classical models discussed herein. In [2] we have presented a novel approach to the quantization of classical models of physical systems whose phase space is the Euclidean space $\mathbb{R}^8$ = [time-space] $\times$ [frequency-wavevector]. The procedure converts functions defined on $\mathbb{R}^8$ into operators within the Hilbert space of signals (analogous to fields) on space-time or, equivalently, in the Hilbert space of their Fourier transforms.

While our approach fundamentally relies on the Weyl-Heisenberg symmetry, a cornerstone in quantum mechanics and quantum optics, an essential ingredient of the construction of the coherent states [3, 4], it can be viewed as non-orthodox. This characterization arises from its departure from conventional quantization methods, notably by omitting the incorporation of the Planck constant. Consequently, there are no associated particles, mass, energy, or momentum attributed to the resulting quantum objects. Our quantization procedure is characterized by an informational essence, stemming from the acknowledgment of the inherent incompleteness present in any classical mathematical model of a physical system. To illustrate this concept, we examine the metric field of general relativity, although the methodology can be extended to other fields, such as the Maxwell electromagnetic field.

These fields traditionally depend on space-time coordinates, but our approach extends their variable domain by incorporating frequencies and wave vectors. The latter can be construed as phenomenological quantities, akin to those introduced in various studies, for example, describing waves in plasmas.

Focusing on general relativity, we demonstrate that the quantization of a metric field satisfying the Einstein equation in empty space yields a regularized version of the field and a corresponding stress-energy tensor. While the existence of the latter may be deemed "fictitious" in some contexts, we argue to the contrary. We posit that its presence is inevitable due to our current lack of an "exact" mathematical model to describe gravitation, possibly providing an informational justification for Modified Newtonian Dynamics (MOND), for example.

As an enduring leitmotif shaping our approach, we aspire to embrace the essence captured in the following quote from [5]:

> *Every field, in our opinion, must therefore adhere to the fundamental principle that singularities of the field are to be excluded.*

In Section 9.2, we embark on a comprehensive exploration of Gabor signal analysis - a method that intricately captures both temporal and frequential facets of a signal. This technique portrays a signal as a function or distribution on the time-frequency plane, laying the foundation for an examination of its profound connection to quantum formalism. Within this section, we unravel the intricate parallels between Gabor signal analysis and key quantum theoretical concepts.

In Section 9.3, we embark on the Gabor quantization of a function defined in the time-frequency phase space, followed by the development of its ensuing semi-classical phase-space portrait.

Expanding upon this groundwork, Section 9.4 delves into the eight-dimensional case, precisely corresponding to the full phase space of electromagnetism or relativity. Our focus extends to considering the metric tensor of general relativity as a tensor-valued signal defined on space-time, providing a bridge between Gabor signal analysis and the rich landscape of higher-dimensional physical models.

The application of this methodology takes center stage with the application of our procedure to the elementary model of free fall in general relativity. Here, we demonstrate how our approach naturally justifies the regularization proposed as an ansatz by Einstein and Rosen in [5]. We then extends our investigation to the well-known Schwarzschild metric tensor, yielding intriguing outcomes in terms of regularization within the framework of our approach.

In the concluding Section 9.5 we provide additional insights into the informational interpretation of the quantum models generated by our approach for describing physical systems.

9.2 From Gabor Signal Analysis to quantum formalism

9.2.1 From Signal Analysis...

The two ingredients of the Gabor analysis [6] of a temporal signal $s(t) \in L^2(\mathbb{R})$ rests upon the two operations of translation and modulation.

One chooses a probe *or window* or *Gaboret* $\psi(t) \in L^2(\mathbb{R})$ which is well localized in time and frequency at once, and which is normalized, $\|\psi\| = 1$. The probe is then translated in time <u>and</u> modulated in frequency, but its size is not modified (in modulus):

$$\psi(t) \to \psi_{b,\omega}(t) = e^{\mathrm{i}\omega t}\,\psi(t-b) \tag{9.1}$$

The time-frequency or *Gabor* or *windowed Fourier* transform of the signal $s(t)$ is then defined as the Hilbertian projection of the signal on the translated-modulated version of the probe:

$$s(t) \to S[s](b,\omega) \equiv S(b,\omega) = \langle \psi_{b,\omega} | s \rangle = \int_{-\infty}^{+\infty} e^{-\mathrm{i}\omega t}\,\overline{\psi(t-b)}\,s(t)\,\mathrm{d}t\,. \tag{9.2}$$

An example of a Gabor (versus Fourier) transform of a signal is shown in Fig. 9.1. The key feature of this time-frequency portrait of the signal is the conservation of its norm, i.e., of its energy which is defined as the square of the norm:

$$\|s\|^2 = \int_{-\infty}^{+\infty} |s(t)|^2\,\mathrm{d}t = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |S(b,\omega)|^2\,\frac{\mathrm{d}b\,\mathrm{d}\omega}{2\pi} \stackrel{\text{def}}{=} \|S\|^2\,. \tag{9.3}$$

Actually this results from the resolution of the identity fulfilled by the continuous non-orthogonal basis made of the set $\{\psi_{b,\omega}\,,\ (b,\omega) \in \mathbb{R}^2\}$ of translated-modulated versions of the probe:

$$\mathbb{1} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\mathrm{d}b\,\mathrm{d}\omega}{2\pi}\,|\psi_{b,\omega}\rangle\,\langle\psi_{b,\omega}| \tag{9.4}$$

Its consequence is the reciprocity (or *reconstruction*) formula:

$$s(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S(b,\omega)e^{i\omega t}\psi(t-b)\,\frac{\mathrm{d}b\,\mathrm{d}\omega}{2\pi}\,. \tag{9.5}$$

9.2.2 ... to Quantum Formalism

The resolution of the identity, akin to (9.4), is the common guideline. Given a measure space (X,μ) and a (separable) Hilbert space $\mathcal{H}$, a *bounded operator-valued* function

$$X \ni x \mapsto \mathsf{M}(x) \text{ acting in } \mathcal{H} \tag{9.6}$$

resolves the identity operator $\mathbb{1}$ in $\mathcal{H}$ with respect to the measure μ if

$$\int_X \mathsf{M}(x)\,\mathrm{d}\mu(x) = \mathbb{1} \tag{9.7}$$

holds in a weak sense ($\sim \langle\psi_1|\psi_2\rangle = \int_X \langle\psi_1|\mathsf{M}(x)|\psi_2\rangle\,\mathrm{d}\mu(x)$ for all $\psi_1,\psi_2 \in \mathcal{H}$).

- In **signal analysis**, *analysis* and *reconstruction* are grounded in the application of the resolution of $\mathbb{1}$ on a signal, i.e., a vector in $\mathcal{H}$

$$\mathcal{H} \ni |s\rangle \stackrel{\text{reconstruction}}{=} \int_X \overbrace{\mathsf{M}(x)|s\rangle}^{\text{analysis}}\,\mathrm{d}\mu(x)\,. \tag{9.8}$$

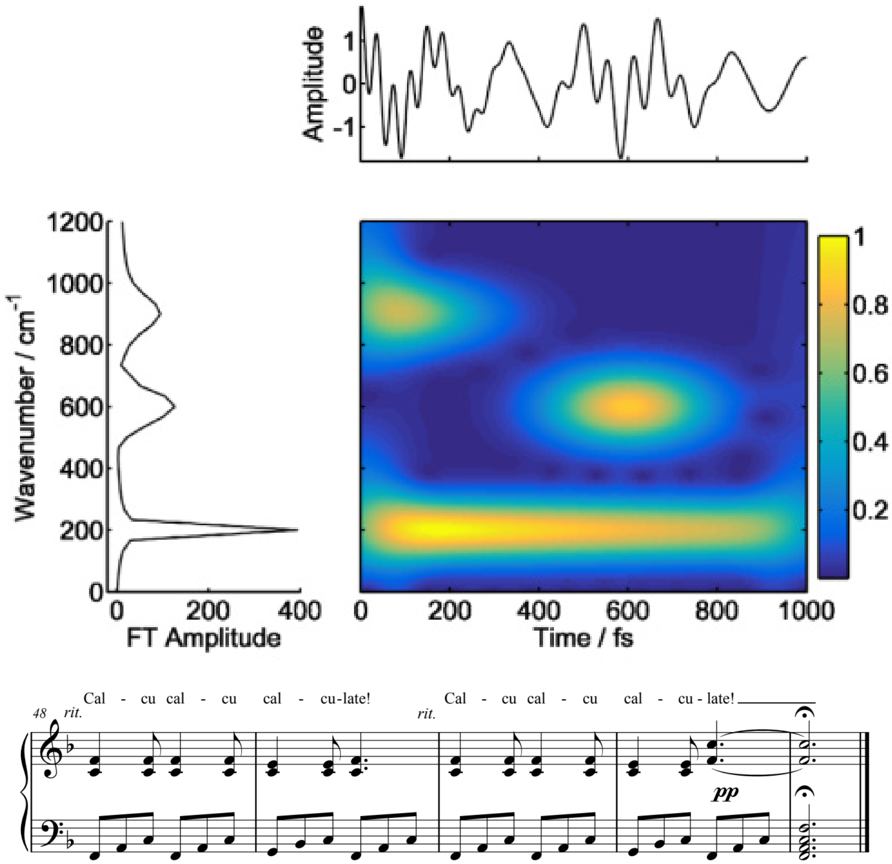

Figure 9.1: **Top**: Presented here is an illustrative instance of the Gabor transform, resembling a time-frequency portrait, applied to the temporal signal depicted at the top. The visualization showcases the modulus of the transform, with color gradation from low values (depicted in blue) to high values (depicted in yellow), spanning the time-frequency inverse half-plane. A comparison is drawn with the modulus of the Fourier transform of the same signal, which is concurrently displayed on the left.
Bottom: Below, we offer a representative example of a musical score, akin to a time-frequency portrait, derived from the sound signal. This portrayal serves as a visual representation of the song's temporal and frequency characteristics.

- In **quantum formalism**, *integral quantization* is grounded in the linear map of a function on X to an operator in $\mathcal{H}$

$$f(x) \mapsto \int_X f(x) \mathsf{M}(x)\, \mathrm{d}\mu(x) = A_f\,, \quad 1 \mapsto \mathbb{1}\,. \tag{9.9}$$

On the flip side, one has the option to delineate the inverse of the aforementioned integral quantization. This semi-classical depiction of the operator A_f or, equivalently, of the original function f, emerges when the family (9.6) of quantizing operators defines a *positive operator-valued measure* (POVM). This implies that the integral quantization (9.9) possesses a profound probability essence, as manifested through this semi-classical portrait. More precisely, if the operators $\mathsf{M}(x)$ in (9.7) are nonnegative, i.e., $\langle\phi|\mathsf{M}(x)|\phi\rangle \geq 0$ for all $x \in X$, one says that they form a POVM on X. If they are further unit trace-class, i.e. $\mathrm{Tr}(\mathsf{M}(x)) = 1$ for all $x \in X$, then the map

$$f(x) \mapsto \text{ semi-classical portrait } \check{f}(x) := \mathrm{Tr}(\mathsf{M}(x)A_f) = \int_X f(x')\,\mathrm{Tr}(\mathsf{M}(x)\mathsf{M}(x'))\,\mathrm{d}\mu(x') \tag{9.10}$$

is a local averaging of the original $f(x)$ (which can very singular, like a Dirac) with respect to the probability distribution on X,

$$x' \mapsto \mathrm{Tr}(\mathsf{M}(x)\mathsf{M}(x'))\,. \tag{9.11}$$

This averaging is in general a regularisation of $f(x)$.

9.3 Gabor positive operator-valued measure (POVM) quantization

9.3.1 Quantization

We now implement the integral quantization defined above to functions or distributions $f(b,\omega)$ defined on the time-frequency plane, precisely by using the resolution of the identity (9.4) provided by the Gabor POVM built from the family of one-rank operators $(b,\omega) \mapsto |\psi_{b\omega}\rangle\langle\psi_{b\omega}|$. Then the quantization of $f(b,\omega)$ reads as:

$$f \mapsto A_f = \int_{\mathbb{R}^2} \frac{\mathrm{d}b\,\mathrm{d}\omega}{2\pi}\, f(b,\omega)\,|\psi_{b\omega}\rangle\langle\psi_{b\omega}|\,. \tag{9.12}$$

The action of A_f on signals is precisely the integral operator

$$(A_f s)(t) = \int_{-\infty}^{+\infty} \mathrm{d}t'\,\mathcal{A}_f(t,t')\,s(t')\,, \tag{9.13}$$

with integral kernel,

$$\mathcal{A}_f(t,t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}b\,\widehat{f}_\omega(b,t'-t)\,\psi(t-b)\,\overline{\psi(t'-b)}\,. \tag{9.14}$$

Here $\widehat{f}_\omega(b,y)$ is the partial Fourier transform with respect to the variable ω:

$$\widehat{f}_\omega(b,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\omega\, f(b,\omega)\,e^{-\mathrm{i}\omega y}\,. \tag{9.15}$$

Applying the above quantization procedure to the time and frequency conjugate variables yields the centered (essentially) self-adjoint time and frequency operators:

$$A_b = T - \langle T\rangle_\psi\,\mathbb{1}\,,\quad (Ts)(t) = ts(t)\,, \tag{9.16}$$

$$A_\omega = \Omega - \langle\Omega\rangle_\psi\,\mathbb{1}\,,\quad (\Omega s)(t) = -\mathrm{i}\,\partial_t s(t)\,, \tag{9.17}$$

where $\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$. Note that $\langle T\rangle_\psi = 0 = \langle\Omega\rangle_\psi$ for any even probe ψ. The "canonical" commutation rule (CCR) is satisfied without calling to the Planck constant $\hbar$, since time and frequency, as physical quantities, have inverse dimension.

$$[A_b, A_\omega] = [T, \Omega] = \mathsf{i}\,\mathbb{1} \tag{9.18}$$

At this point we should keep in mind that the CCR $[A, B] = \mathsf{i}\mathbb{1}$ holds true with self-adjoint A, B with common domain only if both have continuous spectrum $(-\infty, +\infty)$.

Concerning the square of time and frequency variables we obtain:

$$A_{b^2} = (T - \langle T\rangle_\psi)^2 + \Delta_\psi^2 T\,\mathbb{1}\,, \tag{9.19}$$

$$A_{\omega^2} = (\Omega - \langle\Omega\rangle_\psi)^2 + \Delta_\psi^2 \Omega\,\mathbb{1}\,. \tag{9.20}$$

It results the Fourier "uncertainty" relation:

$$\Delta_s T\,\Delta_s\Omega \geq \frac{1}{2}\,, \quad \Delta_s A := \sqrt{\langle s|A^2|s\rangle - (\langle s|A|s\rangle)^2} \tag{9.21}$$

For instance, with the choice of the Gaussian probe with width σ,

$$\psi(t) = G_\sigma(t) = \frac{1}{\pi^{1/4}\sqrt{\sigma}} e^{-\frac{t^2}{2\sigma^2}}\,, \tag{9.22}$$

these operators read:

$$A_{b^2} = T^2 + \frac{\sigma^2}{2}\,\mathbb{1}\,, \quad A_{\omega^2} = \Omega^2 + \frac{\sigma^2}{2}\,\mathbb{1}\,. \tag{9.23}$$

The quantization of the separable function $f(b,\omega) = u(b)v(\omega)$ acts on a signal $s(t)$ as a combination of convolution and multiplication:

$$(A_{uv}s)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}b\,\psi(b)\,u(t-b)\,\left(\overline{\widetilde{\psi}}_b \hat{\overline{v}} * s\right)(t)\,,\ \psi_b(t) := \psi(t-b)\,,\ \widetilde{\psi}(t) := \psi(-t)\,. \tag{9.24}$$

For the monovariable temporal function $f(b,\omega) = u(b)$, one gets the multiplication operator:

$$(A_{u(b)}s)(t) = \left(u * |\psi|^2\right)(t)\,s(t)\,. \tag{9.25}$$

In turn, for the monovariable frequential function $f(b,\omega) = v(\omega)$, one gets the convolution

$$(A_{v(\omega)}s)(t) = \frac{1}{\sqrt{2\pi}} \left[\left(R_{\psi\psi}\hat{\overline{v}}\right) * s\right](t)\,, \tag{9.26}$$

where $R_{\psi\psi}$ is the autocorrelation of the probe, i.e., the correlation of the probe with a delayed copy of itself as a function of the delay,.

$$R_{\psi\psi}(t) := \int_{-\infty}^{+\infty} \mathrm{d}t'\,\psi(t')\,\overline{\psi(t'-t)} = \left(\psi * \overline{\widetilde{\psi}}\right)(t) = \overline{\widetilde{R}}_{\psi\psi}(t)\,. \tag{9.27}$$

9.3.2 Semi-classical portraits

In the case of the Gabor integral quantization, the semi-classical portrait (or *lower symbol*) of A_f is given by:

$$\check{f}(b,\omega) = \int_{\mathbb{R}^2} \frac{\mathrm{d}b'\,\mathrm{d}\omega'}{2\pi}\, f(b',\omega')\, |\langle \psi_{b\omega}|\psi_{b'\omega'}\rangle|^2 \tag{9.28}$$

Note that $(b',\omega') \mapsto |\langle \psi_{b\omega}|\psi_{b'\omega'}\rangle|^2/2\pi$ is a probability distribution on the phase space. Hence regularity properties of $\check{f}$ depend on those of the probe. For $f(b,\omega) = u(b)$ one gets the double or simple convolution:

$$\check{u}(b,\omega) \equiv \check{u}(b) = \left(u * \left(|\psi|^2 * |\widetilde{\psi}|^2\right)\right)(b) = \left(u * R_{|\psi|^2|\psi|^2}\right)(b)\,. \tag{9.29}$$

where we note the appearance of the autocorrelation of the probability distribution $t \mapsto |\psi(t)|^2$ on the temporal axis. For the simplest cases time and time squared, these formulas simplify to:

$$\check{b} = b - \langle b\rangle_{R_{|\psi|^2|\psi|^2}}\,, \quad \check{b^2} = \left(b - \langle b\rangle_{R_{|\psi|^2|\psi|^2}}\right)^2 + \sigma^2_{R_{|\psi|^2|\psi|^2}}(b)\,, \tag{9.30}$$

where $\langle s\rangle_p$ is the expected value of s and $\sigma_p^2(s(t))$ is its variance w.r.t. the probability distribution p.

An analogous formula holds (in the Fourier side) for $f(b,\omega) = v(\omega)$:

$$\check{v}(b,\omega) \equiv \check{v}(\omega) = \overline{\mathcal{F}}\left[\hat{v}\left(\overline{\widetilde{\psi}} * \psi\right)^2\right](\omega)\,, \quad \overline{\mathcal{F}} := \mathcal{F}^{-1}\,. \tag{9.31}$$

By choosing again the Gaussian probe (9.22), we find that the semi-classical portrait is a double Gaussian convolution

$$\check{f}(b,\omega) = \int_{\mathbb{R}^2} \frac{\mathrm{d}b'\,\mathrm{d}\omega'}{2\pi}\, f(b',\omega')\, e^{-\dfrac{(b-b')^2}{2\sigma^2}}\, e^{-\dfrac{\sigma^2(\omega-\omega')^2}{2}} \tag{9.32}$$

One should notice the absence of classical limit at $\sigma \to 0$ or $\sigma \to \infty$, which is one more illustration of the time-frequency uncertainty principle.

The semi-classical portrait of a separable functions $f(b,\omega) = u(b)v(\omega)$ remains separable:

$$\check{f}(b,\omega) = \left(u * G^2_{\sqrt{2}\sigma}\right)(b)\,\left(v * G^2_{\sqrt{2}/\sigma}\right)(\omega)\,. \tag{9.33}$$

In particular:

$$\check{u}(b) = \left(u * G^2_{\sqrt{2}\sigma}\right)(b)\,, \quad \check{v}(\omega) = \left(v * G^2_{\sqrt{2}/\sigma}\right)(\omega)\,. \tag{9.34}$$

Here the classical limit exists separately, and the convergence is simple at least for regular enough u and v:

$$\check{u}(b) \underset{\sigma\to 0}{\to} u(b)\,, \quad \check{v}(\omega) \underset{\sigma\to\infty}{\to} v(\omega)\,. \tag{9.35}$$

Hence, for the simplest cases,

$$\check{b} = b\,, \quad \check{b^2} = b^2 + \sigma^2\,, \quad \check{\omega} = \omega\,, \quad \check{\omega^2} = \omega^2 + \frac{1}{\sigma^2}\,. \tag{9.36}$$

Comparing with (9.16), (9.17), and (9.23) one observes that the original time and frequency variables remain unchanged whereas their quadratic expressions reflect the uncertainty principle in the time-frequency plane and the singular limits at $\sigma \to 0$ (for $\check{\omega}$) and at $\sigma \to \infty$ (for $\check{b}$).

9.3.3 The rôle of the Weyl-Heisenberg group and its representations

The outcomes illustrated earlier can be thoroughly elucidated through the lens of the fundamental Weyl-Heisenberg symmetry that underlies both Gabor analysis and quantum mechanics. It is imperative to recall that the Weyl-Heisenberg group, denoted as G_{WH}, is precisely defined as:

$$G_{\mathrm{WH}} = \{g = (\varsigma, b, \omega)\,,\, \varsigma \in \mathbb{R}\,,\, (b,\omega) \in \mathbb{R}^2\}\,, \tag{9.37}$$

with neutral element: $(0,0,0)$, and

$$g_1 g_2 = \left(\varsigma_1 + \varsigma_2 + \frac{1}{2}(\omega_1 b_2 - \omega_2 b_1)\,, b_1 + b_2,\ \omega_1 + \omega_2\right)\,, \quad g^{-1} = (-\varsigma, -b, -\omega)\,. \tag{9.38}$$

The rôle of the Weyl-Heisenberg group played in underlying the Gabor transform is understood through its unitary irreducible representation (UIR). As a result of the von-Neumann uniqueness theorem, any infinite-dimensional UIR, U, of G_{WH} is characterized by a real number $\lambda \neq 0$ (there is also the degenerate, one-dimensional, UIR corresponding to $\lambda = 0$). If the Hilbert space carrying the UIR is the space of finite-energy signals $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}t)$, the representation operators are defined by ($\lambda = 1$):

$$U(\varsigma, b, \omega) = e^{\mathrm{i}\varsigma} e^{-\mathrm{i}\omega b/2}\, e^{\mathrm{i}\omega T}\, e^{-\mathrm{i}b\Omega} = e^{\mathrm{i}\varsigma}\, e^{\omega T - b\Omega} \equiv e^{\mathrm{i}\varsigma}\, U(b,\omega)\,. \tag{9.39}$$

Now let us pick a unit trace bounded operator $\mathfrak{Q}_0$, and define its Weyl-Heisenberg displaced version as

$$\mathfrak{Q}(b,\omega) = U(b,\omega)\mathfrak{Q}_0 U(b,\omega)^\dagger\,. \tag{9.40}$$

One proves that the family of unit trace bounded operator-valued function $\mathfrak{Q}(b,\omega)$ on the time-frequency plane $\mathbb{R}^2 = \{(b,\omega)\}$ equipped with the invariant measure $\mathrm{d}b\,\mathrm{d}\omega/2\pi$ resolves the identity operator in the Hilbert space $\mathcal{H}$ of signals:

$$\int_{\mathbb{R}^2} \mathfrak{Q}(b,\omega)\,\frac{\mathrm{d}b\,\mathrm{d}\omega}{2\pi} = \mathbb{1}\,. \tag{9.41}$$

The Gabor quantization based on the probe $\psi \in \mathcal{H}$ corresponds to the particular choice $\mathfrak{Q}_0 = |\psi\rangle\langle\psi|$. As previously explained, the relation (9.41) allows the integral quantization of functions on the time-frequency plane:

$$f(b,\omega) \mapsto A_f = \int_{\mathbb{R}^2} f(b,\omega)\mathfrak{Q}(b,\omega)\,\frac{\mathrm{d}b\,\mathrm{d}\omega}{2\pi}\,. \tag{9.42}$$

This quantization is Weyl-Heisenberg covariant:

$$U(b_0,\omega_0)\,A_f\,U(b_0,\omega_0)^\dagger = A_{\mathcal{T}(b_0,\omega_0)f}\,, \quad (\mathcal{T}(b_0,\omega_0)f)\,(b,\omega) = f\,(b - b_0, \omega - \omega_0)\,. \tag{9.43}$$

The following equivalent form of the integral quantization (9.42) is useful for pratical calculations:

$$A_f = \int_{\mathbb{R}^2} U(b,\omega)\,\overline{\mathfrak{F}_{\mathfrak{s}}}[f](b,\omega)\,\Pi(b,\omega)\,\frac{\mathrm{d}b\,\mathrm{d}\omega}{2\pi}\,, \quad \Pi(b,\omega) = \mathrm{Tr}\,(U(-b,-\omega)\mathfrak{Q}_0)\,, \tag{9.44}$$

with $\mathfrak{F}_{\mathfrak{s}}[f](b,\omega) = \int_{\mathbb{R}^2} e^{-\mathrm{i}(b\omega'-b'\omega)}\, f(b',\omega')\, \frac{\mathrm{d}b'\,\mathrm{d}\omega'}{2\pi}$ (symplectic Fourier transform).

The quantum phase space portrait of A_f (or f) is the map:

$$f(b,\omega) \mapsto \check{f}(b,\omega) = \mathrm{Tr}\left(\mathfrak{Q}(b,\omega) A_f\right) = \int_{\mathbb{R}^2} \mathrm{Tr}\left(\mathfrak{Q}(b,\omega)\,\mathfrak{Q}(b',\omega')\right)\, f(b',\omega') \frac{\mathrm{d}b'\,\mathrm{d}\omega'}{2\pi}\,. \tag{9.45}$$

Note its alternative expression:

$$\begin{aligned} \check{f}(b,\omega) &= \int_{\mathbb{R}^2} \mathfrak{F}_{\mathfrak{s}}\left[\Pi\,\widetilde{\Pi}\right](b'-b,\omega'-\omega)\, f(b',\omega')\, \frac{\mathrm{d}b'\,\mathrm{d}\omega'}{2\pi} \\ &= \int_{\mathbb{R}^2} \mathfrak{F}_{\mathfrak{s}}\left[\Pi\right] * \mathfrak{F}_{\mathfrak{s}}\left[\widetilde{\Pi}\right](b'-b,\omega'-\omega)\, f(b',\omega')\, \frac{\mathrm{d}b'\,\mathrm{d}\omega'}{4\pi^2}\,, \end{aligned} \tag{9.46}$$

where $\widetilde{\Pi}(b,\omega) = \Pi(-b,-\omega)$. In the case where the symplectic transform $\mathfrak{F}_{\mathfrak{s}}\left[\Pi\right]$ is non-negative, i.e., is a probability distribution on the time-frequency plane, the convolution

$$\mathfrak{F}_{\mathfrak{s}}\left[\Pi\right] * \mathfrak{F}_{\mathfrak{s}}\left[\widetilde{\Pi}\right] \tag{9.47}$$

is the probability distribution for the difference of two vectors in the time-frequency plane, viewed as independent random variables, and fits to the abelian and homogeneous structure of the time-frequency plane (choice of origin is arbitrary!).

9.4 Eight-dimensional case and general relativity

9.4.1 Gabor material

We now extend the above Gabor (Weyl-Heisenberg covariant) integral quantization of functions on the two-dimensional time-frequency phase space to functions $f(x,k)$ on the phase space $\{(x,k)\in\mathbb{R}^8\}$, where $x=(x^0,\vec{x})=(x^\mu)$ is a time-space variable, and $k=(k^0,\vec{k})=(k^\mu)$ is a frequency–wave-vector variable. We adopt the Minkowskian dot product for the eight-vectors $(x,k)\cdot(x',k') = x\cdot x' + k\cdot k' = x^0x'^0 - \vec{x}\cdot\vec{x}' + k^0k'^0 - \vec{k}\cdot\vec{k}'$ and for the dot product $x\cdot k = x^0k^0 - \vec{x}\cdot\vec{k}$.

Following the same procedure as above, the quantization of a function on $\mathbb{R}^8$ is based on the overcompleteness of the family of translated and modulated unit-norm probes $\psi_{x,k}(y) = e^{\mathrm{i}k\cdot y}\psi(y-x)$:

$$f(x,k) \mapsto A_f = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^8} \mathrm{d}^4x\,\mathrm{d}^4k\, f(x,k)\, \left|\psi_{x,k}\right\rangle\left\langle\psi_{x,k}\right|\,. \tag{9.48}$$

The operator A_f linearly acts in the Hilbert space of finite-energy signals $s(x)\in\mathcal{H} = L^2(\mathbb{R}^4,\mathrm{d}^4x)$ as:

$$s(x) \mapsto (A_f s)(x) = \int_{\mathbb{R}^4} \mathrm{d}^4x'\,\mathcal{A}_f(x,x')\,s(x')\,, \tag{9.49}$$

with integral kernel given by

$$\mathcal{A}_f(x,x') = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \mathrm{d}^4y\,\widehat{f}_k(y,x'-x)\,\psi(x-y)\,\overline{\psi(x'-y)}\,. \tag{9.50}$$

In this expression, $\widehat{f}_k(x,y)$ denotes the partial Fourier transform with respect to the four-vector variable k:

$$\widehat{f}_k(x,y) = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \mathrm{d}^4k\, f(x,k)\, e^{-\mathrm{i}k\cdot y}\,. \tag{9.51}$$

The Gabor semi-classical phase-space portrait of A_f is given by

$$\check{f}(x,k) = \int_{\mathbb{R}^8} \frac{\mathrm{d}^4x'\,\mathrm{d}^4k'}{(2\pi)^4}\, f(x',k')\,|\langle\psi_{x,k}|\psi_{x',k'}\rangle|^2\,. \tag{9.52}$$

For fields $u(x)$ on space-time, like the metric field of GR one gets the multiplication operator

$$(A_u s)(x) = \left(u * |\psi|^2\right)(x)\, s(x)\,, \tag{9.53}$$

and its semi-classical portrait is given by

$$\check{u}(x) = \left(u * \left(|\psi|^2 * |\widetilde{\psi}|^2\right)\right)(x) = \left(u * R_{|\psi|^2|\psi|^2}\right)(x)\,, \quad \widetilde{\psi}(x) = \psi(-x)\,. \tag{9.54}$$

Here convolution and autocorrelation are defined for functions on $\mathbb{R}^4$ equipped with the its Lebesgue measure d^4x:

$$(f*g)(x) = \int_{\mathbb{R}^4} \mathrm{d}^4y\, f(x-y)\, g(y)\,, \tag{9.55}$$

$$R_{\psi\psi}(x) = \int_{\mathbb{R}^4} \mathrm{d}^4x'\,\psi(x')\,\overline{\psi(x'-x)} = \left(\psi * \overline{\widetilde{\psi}}\right)(x)\,. \tag{9.56}$$

This quantization procedure endows space-time variables with the status of quantum observables. The latter act on the space of signals as multiplication operators:

$$x^\mu \mapsto A_{x^\mu} = X^\mu - \langle X^\mu\rangle_\psi\,\mathbb{1}\,, \quad (X^\mu s)(x) = x^\mu s(x)\,. \tag{9.57}$$

The corresponding semi-classical portrait is given by:

$$\widecheck{x^\mu} = x^\mu - \langle x^\mu\rangle_{R_{|\psi|^2|\psi|^2}}\,. \tag{9.58}$$

With the choice of the multivariate Gaussian probe

$$G_{\boldsymbol{\sigma}}(x) = \prod_{\mu=0,1,2,3} G_{\sigma_\mu}(x^\mu)\,, \quad G_\sigma(u) = \frac{1}{\pi^{1/4}\sqrt{\sigma}}e^{-\frac{u^2}{2\sigma^2}}\,, \tag{9.59}$$

the quantum version and the phase portrait of a field $u(x)$ read:

$$(A_u s)(x) = \left(u * G^2_{\boldsymbol{\sigma}}\right)(x)\, s(x)\,, \tag{9.60}$$

$$\check{u}(x) = \left(u * \left(G^2_{\boldsymbol{\sigma}} * G^2_{\boldsymbol{\sigma}}\right)\right)(x) = \left(u * G^2_{\sqrt{2}\boldsymbol{\sigma}}\right)(x)\,. \tag{9.61}$$

Hence, through the map $g_{\mu\nu}(x) \mapsto A_{g_{\mu\nu}}$, the above formalism yields a quantum version of the metric fields $(g_{\mu\nu}(x))$ of general relativity appearing in the genuine space-time metric $\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu$, where one understands the adjective "genuine" as being the starting point for any, regular or singular, global or local, change of coordinates. The subsequent semi-classical portrait $\check{g}_{\mu\nu}(x)$ represents a regularisation of the metric field which depends of the choice of the probe, e.g., the four-width $\boldsymbol{\sigma}$ when the probe is Gaussian.

9.4.2 Elementary examples with Gaussian probes

As a trivial example, the quantization of the Minkowskian metric in Cartesian coordinates,

$$(g_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$$

yields $\left(A_{g_{\mu\nu}}\right) = \mathrm{diag}(\mathbb{1},-\mathbb{1},-\mathbb{1},-\mathbb{1})$. Now, an intriguing question arises when we opt to quantize this metric while expressing it in terms of singular cylindrical coordinates

$$(g^{\mathrm{cyl}}_{\mu\nu}(x^0,\rho,\theta,x^3) = \mathrm{diag}(1,-1,-\rho^2,-1)\,,$$

with $\boldsymbol{\sigma} = (\sigma_0,\sigma,\sigma,\sigma_3)$. We obtain:

$$\begin{aligned} \left(A_{g_{\mu\nu}}\right) &= \mathrm{diag}(\mathbb{1},-\mathbb{1},-(\hat{\rho}^2+\sigma^2\mathbb{1}),-\mathbb{1})\,, \quad \hat{\rho}^2 = \left(X^1\right)^2 + \left(X^2\right)^2\,, \\ \widetilde{\mathrm{d}s^2} &= \mathrm{d}{x_0}^2 - \mathrm{d}\rho^2 - (\rho^2+2\sigma^2)\mathrm{d}\theta^2 - \mathrm{d}{x_3}^2\,. \end{aligned} \tag{9.62}$$

Similarly, for the (singular) spherical coordinates

$$(g^{\mathrm{spher}}_{\mu\nu}(x^0,r,\theta,\phi) = \mathrm{diag}(1,-1,-r^2,-r^2\sin^2\theta \equiv -\rho^2)\,,$$

with $\boldsymbol{\sigma} = (\sigma_0,\sigma,\sigma,\sigma)$, we obtain:

$$\begin{aligned} \left(A_{g_{\mu\nu}}\right) &= \mathrm{diag}\left(\mathbb{1},-\mathbb{1},-\left(\hat{r}^2+\frac{3}{2}\sigma^2\mathbb{1}\right),-(\hat{\rho}^2+\sigma^2)\mathbb{1}\right)\,, \\ \widetilde{\mathrm{d}s^2} &= \mathrm{d}{x_0}^2 - \mathrm{d}r^2 - \left(r^2+3\sigma^2\right)\mathrm{d}\theta^2 - (\rho^2+2\sigma^2)\mathrm{d}\phi^2\,, \end{aligned} \tag{9.63}$$

with $\hat{\rho}^2 = \left(X^1\right)^2 + \left(X^2\right)^2$. One should be aware that the simple adding of a positive constant to ρ^2 (cylindrical case) and to ρ^2 and r^2 (spherical case) creates non-trivial changes to the Euclidean geometry. As a matter of fact, let us consider the Euclidean plane with polar coordinates (r,θ), whose singularity lies at the origin. One finds [7] that the Gabor regularisation of $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \mapsto \mathrm{d}s^2 = \mathrm{d}r^2 + (r^2+r_0^2)\mathrm{d}\theta^2$ yields geodesics involving Jacobi elliptic functions:

$$\frac{r}{r_0} = \pm\sqrt{\frac{1}{k^2\,\mathrm{sn}^2(\theta-\theta_0|k)} - 1}\,.$$

Examples of such geodesics are shown in Fig. 9.2.

9.4.3 Gabor quantization and semi-classical portraits of metric fields

If the metric fields $g_{\mu\nu}(x)$ are known solutions of the Einstein equations for a given tensor energy density

$$R_{\mu\nu} - \frac{1}{2}R = \kappa\, T_{\mu\nu}\,, \quad \kappa = \frac{8\pi G}{c^4}\,, \tag{9.64}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor (symmetric second-degree tensor that depends on only the metric fields and its first and second derivatives), R is the scalar curvature, and

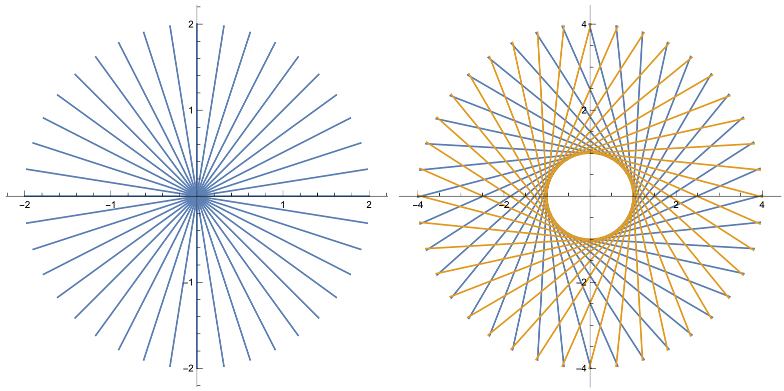

Figure 9.2: Examples of geodesics for the Gabor regularisation of the Euclidean plane metric in polar coordinates.

the $T_{\mu\nu}$'s are the components of the stress-energy tensor. We remind that $(T_{\mu\nu})$ describes the density and flux of energy and momentum in spacetime, generalizing the stress tensor of Newtonian physics. It is an attribute of matter, radiation, and non-gravitational force fields. The respective Gabor-ψ regularised versions of the $g_{\mu\nu}(x)$'s give rise to the modified tensor energy density $\breve{T}_{\mu\nu}$ through the equations:

$$\breve{T}_{\mu\nu} = \frac{1}{\kappa}\left(\breve{R}_{\mu\nu} - \frac{1}{2}\breve{R}\right) . \tag{9.65}$$

This approach provides an avenue to engage with smoothed variations of the metric field, introducing a probabilistic dimension to the smoothing process. Consequently, the concept of empty space takes on a mathematically idealistic nature, as any quantization akin to the Gabor type introduces a fictitious matter, however minute it might be. This emergence of this entity stems from the inherent lack of information encoded in the parameters of the probing function ψ.

A critical aspect for examination pertains to the physical significance embedded in the set of quantization parameters, such as the ensemble $\boldsymbol{\sigma}$ comprising Gaussian widths. The probabilistic character of these parameters should mirror our inherent inability to achieve exactness in terms of information about the observed system, given the constraints of available data and interpretative models.

Furthermore, it is essential to explore the limits inherent in these parameters. Beyond certain threshold values, the mathematical model loses its physical relevance. In other words, these limits signify a point where measurements or observations cease to make meaningful sense within the context of the model.

9.4.4 Uniformly accelerated reference system (free fall)

The metric field for free fall reads as

$$\mathrm{d}s^2 = \alpha^2 x_1^2 \mathrm{d}x_0^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2 \,.$$

For $x_1 \neq 0$ this leads to the Einstein field equations $R_{\mu\nu} = 0$ The restriction $x_1 \neq 0$ is necessary since the Ricci tensor is indeterminate on the hyperplane $x_1 = 0$. In view of regularisation at this singular manifold, Einstein and Rosen introduced a small constant ς $\mathrm{d}s^2 = \left(\alpha^2 x_1^2 + \varsigma\right) \mathrm{d}x_0^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2$. This ansatz leads to the emergence of a fictitious stress-energy tensor $\breve{T}_{\mu\nu}$ within the gravitational equation with a specified source:

$$\breve{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \breve{R} = \kappa \breve{T}_{\mu\nu} \,, \quad \kappa = \frac{8\pi G}{c^4} \,, \tag{9.66}$$

with nonvanishing components $\breve{T}_{22} = \breve{T}_{33} = -\frac{\alpha^2 \varsigma}{\kappa} \left(\alpha^2 x_1^2 + \varsigma\right)^{-2}$ which cancels at $\varsigma = 0$, as expected. This stress-energy tensor is a pure pressure in the directions 2 and 3.

Actually, the Einstein-Rosen ansatz naturally emerges from the Gabor quantization procedure applied to the original metric field and the resulting semi-classical portraits. With the Gaussian probe we get for the metric operators

$$\left(A_{g_{\mu\nu}}\right) = \mathrm{diag}\left(\alpha^2 X_1^2 + \alpha^2 \frac{\sigma_1^2}{2} \mathbb{1}, -\mathbb{1}, -\mathbb{1}, -\mathbb{1}\right) \,, \quad X_1 s(x) = x_1 s(x) \,,$$

where σ_1 is the Gaussian width for the variable x_1, and for their semi-classical portrait,

$$\left(\breve{g}_{\mu\nu}\right) = \mathrm{diag}\left(\alpha^2 x_1^2 + \alpha^2 \sigma_1^2, -1, -1, -1\right) \,. \tag{9.67}$$

Thus the Einstein-Rosen parameter can be identified as

$$\varsigma = \alpha^2 \sigma_1^2 \tag{9.68}$$

and is interpreted as proportional to the Gaussian variance for the variable x_1. The nonvanishing components of regularising stress-energy tensor read:

$$\breve{T}_{22} = \breve{T}_{33} = -\frac{\sigma_1^2}{\kappa} \left(x_1^2 + \sigma_1^2\right)^{-2} \,, \tag{9.69}$$

9.4.5 The example of the Schwarzschild metric field

The well known Schwarzschild solution [8] for the static spherically symmetric field produced by a spherical symmetric body at rest is given by

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\phi^2 \\ &\equiv U \mathrm{d}t^2 - V \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\phi^2 \,, \end{aligned} \tag{9.70}$$

with appropriate units. Note that the metric for the de Sitter space-time with horizon has the same form, just replace $2m/r$ with $\Lambda r^2/3$ in g_{00} and g_{rr}, where Λ is the positive cosmological constant. The cosmological horizon holds at $r = \sqrt{3/\Lambda}$.

In view of regularisation (to a certain extent) we pick a regular isotropic ψ such that $|\psi|^2(x) \equiv |\psi|^2(t, r)$ and, consequently, $R_{|\psi|^2|\psi|^2}(r)$ are probability distributions. Then the Gabor quantization yields the metric multiplication operators (where $Y_r(r') := Y(r' - r)$ is the Heaviside function, and the integration variable in expected values are primed in order to avoid the confusion with the external variable r)

- For U

$$(A_U s)(x) = \left(1 - \frac{2m}{r} + 2m \left\langle Y_r(r')\left(\frac{1}{r} - \frac{1}{r'}\right)\right\rangle_{|\psi|^2}\right) s(x) \equiv \widetilde{U}_{|\psi|^2}(r)s(x)$$

- For V, after regularisation of logarithmic singularity through Cauchy principal value,

$$(A_V s)(x) = \left(1 + \frac{2m}{r} + \frac{2m}{r}\left\langle \frac{m}{r'} \ln \frac{|r+r'-2m|}{|r-r'-2m|}\right\rangle_{|\psi|^2} + \right.$$
$$\left. +2m \left\langle Y_r(r')\left(\frac{1}{r'} - \frac{1}{r} + \frac{m}{rr'} \ln \frac{|r-r'-2m|}{|r'-r-2m|}\right)\right\rangle_{|\psi|^2}\right) s(x) \equiv \widetilde{V}_{|\psi|^2}(r)s(x)\,.$$

- For other terms

$$(A_{r^2} s)(x) = \left(r^2 + \langle r^2\ \rangle_{|\psi|^2}\right) s(x)\,, \quad (A_{r^2\sin^2\theta} s)(x) = \left(r^2 \sin^2\theta + \frac{2}{3}\langle r^2\ \rangle_{|\psi|^2}\right) s(x)\,.$$

The corresponding semi-classical portraits have similar expressions provided that we replace $|\psi|^2$ with $R_{|\psi|^2|\psi|^2}$, both denoted by p. With $p = |\psi|^2$ or $R_{|\psi|^2|\psi|^2}$, the regularised versions read

$$\widetilde{U}_p(r) = 1 - \frac{2m}{r}\left\langle \mathbb{1}_{[0,r]}(r')\right\rangle_p - 2m\left\langle Y_r(r')\frac{1}{r'}\right\rangle_p\,,$$
$$\widetilde{V}_p(r) = 2 - \widetilde{U}_p(r) + \text{ Logarithmic terms}\,,$$

Hence, the regularisation at classical Schwarzschild radius value $r = 2m$ reads:

$$\widetilde{U}_p(2m) = \left\langle Y_{2m}(r')\left(1 - \frac{2m}{r'}\right)\right\rangle_p > 0\,,$$
$$\widetilde{V}_p(2m) = 1 + \left\langle \mathbb{1}_{[0,2m]}(r')\right\rangle_p + \left\langle Y_{2m}(r')\left(\frac{2m}{r'} + \frac{m}{2r'}\ln\frac{r'}{|r'-4m|}\right)\right\rangle_p > 0\,.$$

Its behaviour at large r is given by:

$$\widetilde{U}_p(r) \underset{r\to\infty}{\to} 1\,, \quad \widetilde{V}_p(r) \underset{r\to\infty}{\to} 1\,.$$

On the other hand, its behaviour at $r = 0$ is given by:

$$\widetilde{U}_p(r) \underset{r\to 0}{\to} 1 - \left\langle \frac{2m}{r'}\right\rangle_p\,, \quad \widetilde{V}_p(r) \underset{r\to 0}{\to} 1 + \left\langle \frac{2m}{r'}\right\rangle_p + 4m^2\left\langle \frac{1}{r'(r'-2m)}\right\rangle_p\,,$$

Here, $\widetilde{U}_p(r)$ is monotone increasing from $U_{\min} = 1 - \left\langle \frac{2m}{r'}\right\rangle_p$ (at which the slope is infinite) to 1. Hence one deals with two cases.

1. $U_{\min} > 0$, i.e., $\left\langle \frac{2m}{r'}\right\rangle_p < 1$. Then the temporal term of the Schwarzschild metric is completely regularised.

2. $U_{\min} \leq 0$, i.e., $\left\langle \frac{2m}{r'} \right\rangle_p \geq 1$. Then there exists a smaller "Schwarzschild radius" $r_{s0} \in (0, 2m)$ for the temporal part, defined by the equation

$$r_{s0} = \frac{2m \left\langle \mathbb{1}_{[0,r_{s0}]}(r') \right\rangle_p}{1 - 2m \left\langle Y_{r_{s0}}(r') \frac{1}{r'} \right\rangle_p} .$$

The situation is less obvious for the radial metric term $\widetilde{V}_p(r)$.

9.5 Conclusion

Any mathematical model designed to describe our physical environment, whether successively or not, inevitably introduces drawbacks, often manifesting as singularities. Embracing the challenge, efforts to salvage the advantageous aspects while discarding problematic elements are encouraged. This practice is notably observed in models within the realm of general relativity.

In this contribution, we have illustrated, using various examples, that the application of the Gabor quantization procedure, followed by the development of a semi-classical portrayal of the original classical model, facilitates a specific form of regularization. The profound implication of this regularization, though often subtle, lies in its informational content, a characteristic especially prominent in natural sciences. The certainty of the model's exactness is perpetually uncertain. The merit of our quantization procedure lies in its ability to acknowledge this inherent uncertainty by incorporating it into the formalism, particularly in the selection of the probe. This approach effectively addresses the unavoidable uncertainties, enriching the model's robustness in navigating the intricacies of the physical environment.

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