Chapter 8

Speed of disentanglement for a two-qubit system staged in a Minkowski space with compact support

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8.1 Introduction

The association between special relativity (SR) and quantum mechanics (QM) was established shortly after its inception. In 1926 Schrödinger worked on it [1], followed by Klein [2], Gordon [3], and Dirac [4], leading to the Klein-Gordon equation for integer spin particles and the Dirac equation for spin 1/2 particles [5]. The ascent of quantized fields in the quantum theory and the axiom of the invariance of Lagrangians by Lorentz transformations followed. However, the purpose of this paper is not to combine non-relativistic QM with SR. Instead, it aims to present an interesting parallelism between the SR and QM, at least in the formal realm. A few years ago, researchers studied the separability-entanglement puzzle of bipartite systems from a new perspective [6, 7]. They viewed it as a trajectory along a flat Minkowski phase space of compact support, which is different from the space and time variables of SR that are defined within the domain $(-\infty, \infty)$. In those studies, a physical object comprising two qubits was selected, and its mixed state was represented by a density operator. The theory proposed a construct that demonstrated similarities to SR, with a Minkowski 4D space constrained into a compact domain. This space allowed for "trajectories" to be drawn through two distinct regions. One region contained world line trajectories that was denominated "separable-like", alluding to the time-like region of SR, the other one, "entangled-like" alludes to the space-like.

In paper [6], various physical systems, such those in [8–19], were studied and analyzed. The trajectories for some of these systems were outlined in the Minkowski phase space. Based on the chosen parameters of the systems under investigation, it was revealed that the trajectories that start within the entangled-like region eventually go asymptotically towards the separable-like one. These trajectories could even cross the "light-like cone", for example, beginning as separable, become entangled, become separable again, and so on for several times.

According to current physical theories, it is impossible for matter or information to travel faster than the speed of light, which is also known as superluminal (faster-than-light) or supercausal transmission. Special relativity says that only particles with zero mass, such as photons and possibly neutrinos, can travel at the speed of light c. Although superluminal motion of any kind of particle contradicts the theory of relativity, nonetheless, in QM some effects suggest otherwise when submitted to sieves. These include the tunneling effect, the non-local spooky action at a distance [20, 21], and the loss of entanglement of a bipartite/multipartite system under the influence of the environment.

In this chapter, I briefly review the case of a two-qubit state and draw a formal comparison between the SR Minkowski space and the QM compact support Minkowski space, borrowing the formalism of the former. In SR Minkowski space trajectories represent physical particles that have mass or are massless. On the other hand, in the QM compact Minkowski space, trajectories only describe separability or entanglement of bipartite quantum systems, they do not provide any evidence of meaningful information transfer. I then discuss a closely related issue, the definition and meaning of velocity and speed of disentanglement for a bipartite system when it evolves over time, transiting from entanglement to separability or vice-versa, strangely referred as "sudden death" in [13, 14].

8.2 Space time geometry: brief reminder

The distance between two points in the 3D physical space is expressed as a quadratic equation in Cartesian coordinates

$$ds^2 = dx^2 + dy^2 + dz^2 = dx^i g_{ij} dx^j \tag{8.1}$$

where $g_{ij} = \delta_{ij}$ specifies the flat Euclidian metric, which is invariant under an orthogonal transformation $\mathbb{R}$ of the coordinates, or rotations in 3D, $dx'^i = R^{ik} dx^k$ $\left(\left(R^T\right)^{li} R^{ik} = \delta_{kl}\right)$, as $ds'^2 = R^{ik} dx^k g_{ij} R^{jl} dx^l = dx^k I_{kl} dx^l = ds^2$, and the unit matrix $I_{kl} = \delta_{kl}$. In special relativity (SR) the matrix in 4D $(3+1)$ that specifies the metric is pseudo-Euclidian, or hyperbolic,

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8.2}$$

where the space-time interval between two events is expressed as a quadratic equation containing four terms, the quadratic infinitesimal geodesic segment is

$$ds^2 = dx_0^2 - d\vec{r}^{\,2} = dx_0^2 - d\vec{r} \cdot d\vec{r} = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dx^\mu g_{\mu\nu} dx^\nu \tag{8.3}$$

that must be invariant by a space-time transformations: $D^{\alpha\mu} g_{\mu\nu} D^{\nu\beta} = g_{\alpha\beta}$, with μ, ν, α, $\beta = 0, 1, 2, 3$, where the index 0 is reserved for time.

For a particle traveling along a segment ds one may define a speed that depends on some internal parameter ϑ,

$$u_g(\vartheta) = \frac{ds}{d\vartheta} = +\sqrt{\left(\frac{dx_0}{d\vartheta}\right)^2 - \frac{dx_i}{d\vartheta}\frac{dx_i}{d\vartheta}} \tag{8.4}$$

If one sets ϑ as time t, then

$$u_g(t) = \sqrt{\left(\frac{cdt}{dt}\right)^2 - \frac{dx_i}{dt}\frac{dx_i}{dt}} = \sqrt{c^2 - \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}} = c\sqrt{1 - \frac{\vec{v}^2}{c^2}}. \tag{8.5}$$

As it is assumed axiomatically that a particle running along a geodesic segment has a positive speed, $u_g(t) > 0$, therefore the particle speed in the 3D should not exceed the speed of light in vacuum $c > v$. Nonetheless, from eq. (8.3) the inequality $ds^2 < 0$ is not ruled out when a particle is assumed to move at virtual superluminal speed, and its trajectory resides within a region called *spacelike*.

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx_0}{dt}\right)^2 - \left(\frac{d\vec{r}}{dt}\right)^2 = c^2 - \vec{v}^2 < 0\,.$$

8.3 The two-qubit state

The most general two-qubit state (pure or mixed after tracing out over a (N-2)D subsystem) in the so-called Fano's form is

$$\hat{\rho} = \frac{1}{2^2}\left(1_1 \otimes 1_2 + 1_1 \otimes \vec{\sigma}_2 \cdot \vec{P}_2 + \vec{\sigma}_1 \cdot \vec{P}_1 \otimes 1_2 + \vec{\sigma}_1 \cdot \overleftrightarrow{M} \cdot \vec{\sigma}_2\right) \tag{8.6}$$

where $\vec{\sigma}$ is a vector whose, x, y, and z components are the Pauli matrices σ_x, σ_y, and σ_z. $\vec{P}_k = \mathrm{Tr}\left(\vec{\sigma}_k \hat{\rho}\right)$ is a polarization vector (PV) associated to each qubit, $k = 1, 2$, $\overleftrightarrow{M}$ is a dyadic operator for the correlation matrix

$$\mathbb{M} = \begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix}, \tag{8.7}$$

(the subscript on the left stands for qubit 1 and the other for qubit 2) with entries $M_{ij} = \langle \sigma_{1,i}\sigma_{2,j} \rangle = \mathrm{Tr}\left(\hat{\rho}\sigma_{1,i}\sigma_{2,j}\right)$ and $|M_{ij}| \leq 1$. A separable two-qubit state has the following properties: (a) the PV's are written as $\vec{P}_\mu = \sum_k p_k \vec{Q}_\mu^{(k)}$, where $\vec{Q}_\mu^{(k)}$ $(\mu = 1, 2)$ is a vector, $\left|\vec{Q}_\mu^{(k)}\right| \leq 1$, the superscript k characterizes a direction in 3D and, (b) the dyadic could be $\overleftrightarrow{M} = \sum_k p_k \vec{Q}_1^{(k)} \vec{Q}_2^{(k)}$, with weights $p_k \in [0, 1]$ and $\sum_k p_k = 1$, the state (8.6) becomes

$$\hat{\rho}^{\mathrm{sep}} = \frac{1}{4}\sum_k p_k \left(1_1 + \vec{Q}_1^{(k)} \cdot \vec{\sigma}_1\right) \otimes \left(1_2 + \vec{Q}_2^{(k)} \cdot \vec{\sigma}_2\right), \tag{8.8}$$

thus, in this state the qubits are not entangled.

8.3.1 Polarization vectors and correlation matrix

The state (8.6) can also be displayed [6, 7] as a matrix,

$$\hat{\rho}=\frac{1}{4}\left(\begin{array}{cc} 1+P_{1,z}+P_{2,z}+M_{zz} & P_{2,x}-iP_{2,y}+M_{zx}-iM_{zy} \\ P_{2,x}+iP_{2,y}+M_{zx}+iM_{zy} & 1+P_{1,z}-P_{2,z}-M_{zz} \\ P_{1,x}+iP_{1,y}+M_{xz}+iM_{yz} & M_{xx}+M_{yy}-i\left(M_{xy}-M_{yx}\right) \\ M_{xx}-M_{yy}+i\left(M_{xy}+M_{yx}\right) & P_{1,x}+iP_{1,y}-M_{xz}-iM_{yz} \end{array}\right.$$

$$\left.\begin{array}{cc} P_{1,x}-iP_{1,y}+M_{xz}-iM_{yz} & M_{xx}-M_{yy}-i\left(M_{xy}+M_{yx}\right) \\ M_{xx}+M_{yy}+i\left(M_{xy}-M_{yx}\right) & P_{1,x}-iP_{1,y}-M_{xz}+iM_{yz} \\ 1-P_{1,z}+P_{2,z}-M_{zz} & P_{2,x}-iP_{2,y}-M_{zx}+iM_{zy} \\ P_{2,x}+iP_{2,y}-M_{zx}-iM_{zy} & 1-P_{1,z}-P_{2,z}+M_{zz} \end{array}\right), \tag{8.9}$$

that depends on fifteen free parameters, with the constraint $\mathrm{Tr}\left(\hat{\rho}\right)^2 \leq 1$. The polarization vectors are

$$\mathbb{P}_1=\left(\begin{array}{ccc} P_{1,x} & P_{1,y} & P_{1,z} \end{array}\right)^{\mathsf{T}}=\left(\begin{array}{ccc} 2\Re\left(\rho_{13}+\rho_{24}\right) & -2\Im\left(\rho_{13}+\rho_{24}\right) & 2\left(\rho_{11}+\rho_{22}\right)-1 \end{array}\right)^{\mathsf{T}} \tag{8.10a}$$

$$\mathbb{P}_2=\left(\begin{array}{ccc} P_{2,x} & P_{2,y} & P_{2,z} \end{array}\right)^{\mathsf{T}}=\left(\begin{array}{ccc} 2\Re\left(\rho_{12}+\rho_{34}\right) & -2\Im\left(\rho_{12}+\rho_{34}\right) & 2\left(\rho_{11}+\rho_{33}\right)-1 \end{array}\right)^{\mathsf{T}}, \tag{8.10b}$$

the superscript $^{\mathsf{T}}$ stands for transposed and matrix (8.7) is

$$\mathbb{M}=\left(\begin{array}{ccc} 2\Re\left(\rho_{14}+\rho_{23}\right) & 2\Im\left(\rho_{23}+\rho_{41}\right) & 2\Re\left(\rho_{13}-\rho_{24}\right) \\ 2\Im\left(\rho_{41}+\rho_{32}\right) & 2\Re\left(\rho_{23}-\rho_{14}\right) & 2\Im\left(\rho_{24}-\rho_{13}\right) \\ 2\Re\left(\rho_{12}-\rho_{34}\right) & 2\Im\left(\rho_{34}-\rho_{12}\right) & 1-2\left(\rho_{22}+\rho_{33}\right) \end{array}\right). \tag{8.11}$$

The reduced density matrix for each qubit only depends on its own polarization vector

$$\begin{aligned} \mathrm{Tr}_2\left(\hat{\rho}\right) &= \hat{\rho}^{(1)}=\left(\begin{array}{cc} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} \end{array}\right)=\frac{1}{2}\left(\begin{array}{cc} 1+P_{1,z} & P_{1,x}-iP_{1,y} \\ P_{1,x}+iP_{1,y} & 1-P_{1,z} \end{array}\right) \\ \mathrm{Tr}_1\left(\hat{\rho}\right) &= \hat{\rho}^{(2)}=\left(\begin{array}{cc} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{array}\right)=\frac{1}{2}\left(\begin{array}{cc} 1+P_{2,z} & P_{2,x}-iP_{2,y} \\ P_{2,x}+iP_{2,y} & 1-P_{2,z} \end{array}\right) \end{aligned}$$

or, $\hat{\rho}^{(k)}=\frac{1}{2}\left(\hat{1}+\vec{\sigma}_k\cdot\vec{P}_k\right)$, $k=1,2$ with $\left|\vec{P}_k\right|\leq 1$, correlation of one qubit with its partner does not show up.

8.3.2 Positivity Partial Transposition (PPT)

Making use of a specific local operation on each qubit in (8.9), a new matrix, $\hat{\rho}_2^T$, comes out, and the Peres-Horodecki criterion (PHC) [8, 9] permits to identify an entangled state whenever the matrix has at least one negative eigenvalue; it hints that the qubits bear some degree of entanglement. The PHC is based on a *positive partial transposition* (PPT) operation, which consists in making a transposition on only one qubit in the original matrix, and to then analyze the positivity the eigenvalues. Symbolically, $\hat{\rho}^T=\left(\hat{1}_1\otimes\hat{T}_2\right)\hat{\rho}\geq 0$, where $\hat{T}_2$ stands for the transposition operation on particle 2. For instance, partitioning matrix (8.9) into four 2×2 sub-blocks, the transposition is done on the diagonal entries within each sub-block, $\rho_{12}\rightleftarrows\rho_{21}$, $\rho_{14}\rightleftarrows\rho_{23}$, $\rho_{32}\rightleftarrows\rho_{41}$, $\rho_{34}\rightleftarrows$

ρ_{43}, which is a *positive map* but not completely positive and for that reason it provides *a necessary and sufficient condition* test for separability, as pointed in [8, 9]. Thence it becomes possible to characterize the separability (or entanglement) in $\hat{\rho}$ using this procedure.

The exchange of position entries in matrix $\hat{\rho}$, expressed in the computational basis, results in matrix $\hat{\rho}^T$, entailing the following changes in the polarization vector $\vec{P}_2$ for qubit 2, and also in the correlation matrix (8.7),

$$P_{2,y} \longrightarrow -P_{2,y}, \tag{8.13a}$$

$$(M_{xy}, M_{yy}, M_{zy}) \longrightarrow -(M_{xy}, M_{yy}, M_{zy}), \tag{8.13b}$$

$\vec{P}_2$ is substituted by a vector which has undergone a partial reflexion operation, the mirror image reflected by the $x-z$ plane, and the same operation applies to the entries of matrix (8.7), where the second subscript y refers to qubit 2. Hence,

$$\mathbb{P}_2 \longrightarrow \mathbb{P}_2^T = \begin{pmatrix} P_{2,x} & -P_{2,y} & P_{2,z} \end{pmatrix}^T, \tag{8.14}$$

and (8.7) become

$$\mathbb{M} \longrightarrow \mathbb{M}^T = \begin{pmatrix} M_{xx} & -M_{xy} & M_{xz} \\ M_{yx} & -M_{yy} & M_{yz} \\ M_{zx} & -M_{zy} & M_{zz} \end{pmatrix}. \tag{8.15}$$

The maps (8.13) cannot be obtained by an unitary transformation, they correspond to a reflection of the Pauli vector $\vec{\sigma}_2$ by the plane $x-z$ in $\mathcal{R}^3$ space. If one considers $\hat{\rho}$ containg information about the physical reality, $\hat{\rho}^T$ stands for the complementary partially hidden reality.

8.3.3 $D-7$ manifold class matrix with seven free parameters

If one particularizes matrix (8.9) as a one that contains seven free parameters, instead of (8.9) that has fifteen, in the D-7 manifold class the description of entanglement and separability become more prominent. Admitting the PV's aligned along the z-direction $\mathbb{P}_k = \begin{pmatrix} 0 & 0 & P_{k,z} \end{pmatrix}^\mathsf{T}$, $k = 1, 2$, and reducing the number of non-null entries (free parameters) in the CM to five,

$$\mathbb{M} = \begin{pmatrix} M_{xx} & M_{xy} & 0 \\ M_{yx} & M_{yy} & 0 \\ 0 & 0 & M_{zz} \end{pmatrix}. \tag{8.16}$$

one gets matrix

$$\hat{\rho} = \frac{1}{4}\left(\begin{matrix} 1+P_{1,z}+P_{2,z}+M_{zz} & 0 \\ 0 & 1+P_{1,z}-P_{2,z}-M_{zz} \\ 0 & M_{xx}+M_{yy}+i\,(M_{yx}-M_{xy}) \\ M_{xx}-M_{yy}+i\,(M_{yx}+M_{xy}) & 0 \end{matrix}\right.$$
$$\left.\begin{matrix} 0 & M_{xx}-M_{yy}-i\,(M_{yx}+M_{xy}) \\ M_{xx}+M_{yy}-i\,(M_{yx}-M_{xy}) & 0 \\ 1-P_{1,z}+P_{2,z}-M_{zz} & 0 \\ 0 & 1-P_{1,z}-P_{2,z}+M_{zz} \end{matrix}\right). \tag{8.17}$$

Henceforth, the set of seven free parameters can now combined to define eight new parameters

$$t_{\pm} = \frac{1 \pm M_{zz}}{2}, \quad u_{\pm} = \frac{P_{1,z} \pm P_{2,z}}{2}, \tag{8.18a}$$

$$v_{\pm} = \frac{M_{xx} \pm M_{yy}}{2}, \quad w_{\pm} = \frac{M_{yx} \pm M_{xy}}{2}. \tag{8.18b}$$

(the digit 1 in $t_{\pm}$ completes the set). The eigenvalues of matrix (8.17) are

$$\lambda_1 = (t_- + X_1)/2,\ \lambda_2 = (t_- - X_1)/2,\ \lambda_3 = (t_+ + X_2)/2,\ \lambda_4 = (t_+ - X_2)/2, \tag{8.19}$$

where

$$X_1^2 = u_-^2 + v_+^2 + w_-^2 \qquad \text{and} \qquad X_2^2 = u_+^2 + v_-^2 + w_+^2, \tag{8.20}$$

are specific quadratic distances in 3D under the Euclidean metric. According to the Peres-Horodecki criterion, the partial transposition on qubit 2 is equivalent in making the following changes

$$(t_{\pm}, u_{\pm}, v_{\pm}, w_{\pm}) \to (t_{\pm}, u_{\pm}, v_{\mp}, w_{\mp}) \tag{8.21}$$

and the eigenvalues of the partially transposed matrix $\hat{\rho}^T$ are

$$\lambda_1^T = \left(t_- + X_1^T\right)/2,\ \lambda_2^T = \left(t_- - X_1^T\right)/2,\ \lambda_3^T = \left(t_+ + X_2^T\right)/2,\ \lambda_4^T = \left(t_+ - X_2^T\right)/2, \tag{8.22}$$

where, differently from the quadratic forms (8.20), one has

$$\left(X_1^T\right)^2 = u_-^2 + v_-^2 + w_+^2, \qquad \text{and} \qquad \left(X_2^T\right)^2 = u_+^2 + v_+^2 + w_-^2. \tag{8.23}$$

Comparing Eqs. (8.23) with Eqs. (8.20), one observes that only parameters v and w have their subscripts signs, $+$ and $-$, interchanged. The set of eigenvalues $\left\{\lambda_i^T\right\}$ can be obtained directly from the set $\{\lambda_i\}$ after doing the changes $P_{2,y} \to -P_{2,y}$ and $M_{ky} \to -M_{ky}$ $(k = x, y, z)$ or equivalently, $\{v_{\pm}, w_{\pm}\} \to \{v_{\mp}, w_{\mp}\}$.

Quadratic distance in a 4D Minkowski geometry

The quadratic distances associated with $\hat{\rho}$ and $\hat{\rho}^T$ are defined as

$$\begin{aligned}
s_1^2 &= t_-^2 - \left(u_-^2 + v_+^2 + w_-^2\right) = t_-^2 - \vec{r}_1^{\,2} = t_-^2 - X_1^2 & (8.24a)\\
\left(s_1^T\right)^2 &= t_-^2 - u_-^2 - v_-^2 - w_+^2 = t_-^2 - \left(\vec{r}_1^{\,T}\right)^2 = t_-^2 - \left(X_1^T\right)^2, & (8.24b)\\
s_2^2 &= t_+^2 - u_+^2 - v_-^2 - w_+^2 = t_+^2 - \vec{r}_2^{\,2} = t_+^2 - X_2^2, & (8.24c)\\
\left(s_2^T\right)^2 &= t_+^2 - u_+^2 - v_+^2 - w_-^2 = t_+^2 - \left(\vec{r}_1^{\,T}\right)^2 = t_+^2 - \left(X_1^T\right)^2, & (8.24d)
\end{aligned}$$

furthermore, the invariance $s_1^2 + s_2^2 = \left(s_1^T\right)^2 + \left(s_2^T\right)^2$ is verified, although each term in the left hand side is non-negative, whereas on the right hand side one of the terms can be negative, which becomes the flag that reveals the existence of entanglement.

Partial positivity conditions

The eigenvalues take values in the interval $[0,1]$, and the following inequalities hold

$$\lambda_1\lambda_2 = s_1^2 \equiv t_-^2 - X_1^2 \geq 0 \quad , \quad \lambda_3\lambda_4 = s_2^2 \equiv t_+^2 - X_2^2 \geq 0. \tag{8.25a}$$

$$\lambda_1^T\lambda_2^T = \left(s_1^T\right)^2 \equiv t_-^2 - \left(X_1^T\right)^2 \geq 0 \quad , \quad \lambda_3^T\lambda_4^T = \left(s_2^T\right)^2 \equiv t_+^2 - \left(X_2^T\right)^2 \geq 0, \tag{8.25b}$$

and, in analogy to SR, one is tempted to consider the parameters $t_\pm$, as “times” whereas $\{u_\pm, v_\pm, w_\pm\}$ get the role of distances in 3D Euclidean space. By its turn s_k become the quadridistance in Minkowski space in 4D. Although that admittance remits to SR, nonetheless, differently from it, the manifold has compact support, $s_i^2 \in [0,1]$. In what follows, using the lexicon of SR, we can say that s_1^2 and s_2^2 are *time-like*. The quadratic distances (8.25a) are always non-negative, they do not provide information about the qubits separability or entanglement.

For the PT state $\hat{\rho}^T$ one has the similar inequalities (8.25b) that could reveal, under the PHC, whether, hidden in $\hat{\rho}$, the qubits are entangled. Nonetheless the inequalities (8.25b) are quite different from (8.25a). If the qubits are, in some degree, entangled for some set of numerical values of the intrinsic parameters of the system, then, one of the quadratic quadridistances ($\left(s_1^T\right)^2$ or $\left(s_2^T\right)^2$) will result negative, and according to the SR terminology it it is said to be “space-like”. Thence, using an adjusted lexicon, it can be said that the squared quadridistance in (8.25b) stem into the *separable-like* region whenever $\left(s_1^T\right)^2 > 0$ and $\left(s_2^T\right)^2 > 0$, or into the *entangled-like* whenever one of the quadratic distances in (8.25b takes a negative value, $\left(s_1^T\right)^2 > 0$ and $\left(s_2^T\right)^2 < 0$, or $\left(s_1^T\right)^2 < 0$ and $\left(s_2^T\right)^2 > 0$). The equality $\left(s_k^T\right)^2 = 0$ stands for the border of separability, which corresponds to the *light-like* cone surface of SR.

Furthermore, since $(s_1)^2 = 4\lambda_1\lambda_2$, and $(s_2)^2 = 4\lambda_3\lambda_4$, as well as $\left(s_1^T\right)^2 = 4\lambda_1^T\lambda_2^T$, and $\left(s_2^T\right)^2 = 4\lambda_3^T\lambda_4^T$, the quadratic distances are invariant under similarity transformations applied on states $\hat{\rho}$ and $\hat{\rho}^T$.

8.4 Velocity of disentanglement

The quadridistances in Eqs. (8.24) are related to matrix (8.17) entries

$$\begin{aligned}
s_1 \quad &\Longrightarrow \quad (t_-, u_-, v_+, w_-) = \left(\frac{1-M_{zz}}{2}, \frac{P_{1,z}-P_{2,z}}{2}, \frac{M_{xx}+M_{yy}}{2}, \frac{M_{yx}-M_{xy}}{2}\right),\\
s_2 \quad &\Longrightarrow \quad (t_+, u_+, v_-, w_+) = \left(\frac{1+M_{zz}}{2}, \frac{P_{1,z}+P_{2,z}}{2}, \frac{M_{xx}-M_{yy}}{2}, \frac{M_{yx}+M_{xy}}{2}\right),\\
s_1^T \quad &\Longrightarrow \quad (t_-, u_-, v_-, w_+) = \left(\frac{1-M_{zz}}{2}, \frac{P_{1,z}-P_{2,z}}{2}, \frac{M_{xx}-M_{yy}}{2}, \frac{M_{yx}+M_{xy}}{2}\right),\\
s_2^T \quad &\Longrightarrow \quad (t_+, u_+, v_+, w_-) = \left(\frac{1+M_{zz}}{2}, \frac{P_{1,z}+P_{2,z}}{2}, \frac{M_{xx}+M_{yy}}{2}, \frac{M_{yx}-M_{xy}}{2}\right),
\end{aligned}$$

and the elements of the sets (t_-, u_-, v_+, w_-) and (t_+, u_+, v_-, w_+) may depend, intrinsically, on the physical system parameters present in the Hamiltonian or in the density operator. Two "velocities"in 3D can be defined,

$$\vec{V}_1 = \frac{d\vec{X}_1}{dt_-} = \left(\frac{du_-}{dt_-}, \frac{dv_+}{dt_-}, \frac{dw_-}{dt_-}\right) \tag{8.27a}$$

$$\vec{V}_2 = \frac{d\vec{X}_2}{dt_+} = \left(\frac{du_+}{dt_+}, \frac{dv_-}{dt_+}, \frac{dw_+}{dt_+}\right). \tag{8.27b}$$

A dynamical system has *time* t (the one measured by clocks) as natural parameter. Thus, for all other parameters fixed, one should express the vectors (8.27) as

$$\left(\frac{du_-}{dt_-}, \frac{dv_+}{dt_-}, \frac{dw_-}{dt_-}\right) \Longrightarrow \left(\frac{\partial u_-/\partial t}{\partial t_-/\partial t}, \frac{\partial v_+/\partial t}{\partial t_-/\partial t}, \frac{\partial w_-/\partial t}{\partial t_-/\partial t}\right)_{\text{fixed parameters}}, \tag{8.28a}$$

$$\left(\frac{du_+}{dt_+}, \frac{dv_-}{dt_+}, \frac{dw_-}{dt_+}\right) \Longrightarrow \left(\frac{\partial u_+/\partial t}{\partial t_+/\partial t}, \frac{\partial v_-/\partial t}{\partial t_+/\partial t}, \frac{\partial w_+/\partial t}{\partial t_+/\partial t}\right)_{\text{fixed parameters}}. \tag{8.28b}$$

and the speeds are

$$V_1 = \left|\vec{V}_1\right| = \left(\left(\frac{du_-}{dt_-}\right)^2 + \left(\frac{dv_+}{dt_-}\right)^2 + \left(\frac{dw_-}{dt_-}\right)^2\right)^{1/2} \tag{8.29a}$$

$$V_2 = \left|\vec{V}_2\right| = \left(\left(\frac{du_+}{dt_+}\right)^2 + \left(\frac{dv_-}{dt_+}\right)^2 + \left(\frac{dw_+}{dt_+}\right)^2\right)^{1/2}. \tag{8.29b}$$

Each speed V_1 (V_2) (for each set of internal parameter) can be plotted in parametric form $V_1(t) \times t_-(t)$ and $V_2(t) \times t_+(t)$. In Minkowski 4D space the quadrispeeds are

$$W_1 = \frac{ds_1}{dt_-} = \sqrt{1 - V_1^2} > 0, \tag{8.30a}$$

$$W_2 = \frac{ds_2}{dt_+} = \sqrt{1 - V_2^2} > 0 \tag{8.30b}$$

and whenever V_1^2, $V_2^2 < 1$ number 1 becomes the maximum speed value, the equivalent to c, the speed of light in vacuum.

The same goes for PT matrix, the "velocities"are

$$\vec{V}_1^T = \frac{d\vec{X}_1^T}{dt_-} = \left(\frac{du_-}{dt_-}, \frac{dv_-}{dt_-}, \frac{dw_+}{dt_-}\right), \tag{8.31a}$$

$$\vec{V}_2^T = \frac{d\vec{X}_2^T}{dt_+} = \left(\frac{du_+}{dt_+}, \frac{dv_+}{dt_+}, \frac{dw_-}{dt_+}\right), \tag{8.31b}$$

and the speeds are

$$V_1^T = \left|\vec{V}_1^T\right| = \left(\left(\frac{du_-}{dt_-}\right)^2 + \left(\frac{dv_-}{dt_-}\right)^2 + \left(\frac{dw_+}{dt_-}\right)^2\right)^{1/2}, \tag{8.32a}$$

$$V_2^T = \left|\vec{V}_2^T\right| = \left(\left(\frac{du_+}{dt_+}\right)^2 + \left(\frac{dv_+}{dt_+}\right)^2 + \left(\frac{dw_-}{dt_+}\right)^2\right)^{1/2}. \tag{8.32b}$$

In Minkowski 4D space,

$$\left(W_1^T\right)^2 = \left(\frac{ds_1^T}{dt_-}\right)^2 = 1-\left(V_1^T\right)^2 \Longrightarrow \frac{ds_1^T}{dt_-} = \sqrt{1-\left(V_1^T\right)^2} \tag{8.33a}$$

$$\left(W_2^T\right)^2 = \left(\frac{ds_2^T}{dt_+}\right)^2 = 1-\left(V_2^T\right)^2 \Longrightarrow \frac{ds_2^T}{dt_+} = \sqrt{1-\left(V_2^T\right)^2}. \tag{8.33b}$$

Factually, the sets (t_-, u_-, v_-, w_+) and (t_+, u_+, v_+, w_-) depend on the parameters of the system, and for a system evolving in time t should be considered. The equivalent to Eqs. (8.28) are

$$\left(\frac{du_-}{dt_-}, \frac{dv_-}{dt_-}, \frac{dw_+}{dt_-}\right)^T \Longrightarrow \left(\frac{\partial u_-/\partial t}{\partial t_-/\partial t}, \frac{\partial v_-/\partial t}{\partial t_-/\partial t}, \frac{\partial w_+/\partial t}{\partial t_-/\partial t}\right)^T_{\text{fixed parameters}}. \tag{8.34a}$$

$$\left(\frac{du_+}{dt_+}, \frac{dv_+}{dt_+}, \frac{dw_-}{dt_+}\right)^T \Longrightarrow \left(\frac{\partial u_+/\partial t}{\partial t_+/\partial t}, \frac{\partial v_+/\partial t}{\partial t_+/\partial t}, \frac{\partial w_-/\partial t}{\partial t_+/\partial t}\right)^T_{\text{fixed parameters}}. \tag{8.34b}$$

For each speed V_1, V_2, the plot can be done as $V_1^T \times t_-$ and $V_2^T \times t_+$ or else, in parametric form $V_1^T(t) \times t_-(t)$ and $V_2^T(t) \times t_+(t)$.

8.5 The Blank-Exner-Werner state

I shall illustrate now the main point of the chapter, namely, how to calculate the speed of disentanglement? To simplify the calculations I borrowed the Blank-Exner-Werner (BEW) [10, 11] state, where the polarization vector and the correlation matrix entries depend only on a single parameter – see the Table bellow.

State	$P_{1,z}$	$P_{2,z}$	M_{xx}	M_{yy}	M_{xy}	M_{yx}	M_{zz}
WBE	0	0	$-x$	$-x$	0	0	$-x$

The state is a mixture of a 2-qubit state (two spin-1/2 singlet state) $|\psi_-\rangle$ balanced with a total stochastic state I,

$$\hat{\rho}(x) = x\,|\psi_-\rangle\langle\psi_-| + \frac{1-x}{4} I\ , \tag{8.35}$$

where the parameter $x \in [0,1]$ is a weight (or probability). In the computational basis the stochastic state is

$$I = |00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 01| + |10\rangle\langle 10| \tag{8.36}$$

where the vectors $|00\rangle\langle 00|$, $|11\rangle\langle 11|$ belong to the "spin-1" triplet, whereas the states

$$|\psi_\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}, \tag{8.37}$$

with subscript $+$ it represents the zero-projection state of the triplet and the sign $-$ labels the singlet state. Inverting the nomenclature $\{|\psi_-\rangle,\ |\psi_+\rangle\} \Longrightarrow \{|01\rangle, |10\rangle\}$, one has

$$|01\rangle = \frac{|\psi_-\rangle + |\psi_+\rangle}{\sqrt{2}},\ |10\rangle = \frac{|\psi_+\rangle - |\psi_-\rangle}{\sqrt{2}}. \tag{8.38}$$

The outer products are,

$$\left|\psi_{\pm}\right\rangle\left\langle\psi_{\pm}\right| = \frac{1}{2}\left(\left|01\right\rangle\left\langle 01\right| + \left|10\right\rangle\left\langle 10\right| \pm \left(\left|01\right\rangle\left\langle 10\right| + \left|10\right\rangle\left\langle 01\right|\right)\right), \tag{8.39}$$

such that

$$\left|01\right\rangle\left\langle 01\right| = \frac{1}{2}\left(\left|\psi_-\right\rangle\left\langle\psi_-\right| + \left|\psi_+\right\rangle\left\langle\psi_+\right| + \left|\psi_-\right\rangle\left\langle\psi_+\right| + \left|\psi_+\right\rangle\left\langle\psi_-\right|\right) \tag{8.40a}$$

$$\left|10\right\rangle\left\langle 10\right| = \frac{1}{2}\left(\left|\psi_+\right\rangle\left\langle\psi_+\right| + \left|\psi_-\right\rangle\left\langle\psi_-\right| - \left|\psi_+\right\rangle\left\langle\psi_-\right| - \left|\psi_-\right\rangle\left\langle\psi_+\right|\right). \tag{8.40b}$$

Summing the above terms one gets

$$\left|01\right\rangle\left\langle 01\right| + \left|10\right\rangle\left\langle 10\right| = \left|\psi_-\right\rangle\left\langle\psi_-\right| + \left|\psi_+\right\rangle\left\langle\psi_+\right| , \tag{8.41}$$

reminding that

$$\left|01\right\rangle = \left|0\right\rangle_1 \otimes \left|1\right\rangle_2 ; \quad \left|01\right\rangle\left\langle 01\right| = \left(\left|0\right\rangle_1 \otimes \left|1\right\rangle_2\right)\left({}_1\left\langle 0\right| \otimes {}_2\left\langle 1\right|\right).$$

Writing the outer products in a blend of both basis one gets

$$\begin{aligned}\hat{\rho}(x) &= x\left|\psi_-\right\rangle\left\langle\psi_-\right| + \frac{1-x}{4}\left(\left|00\right\rangle\left\langle 00\right| + \left|11\right\rangle\left\langle 11\right| + \left|\psi_-\right\rangle\left\langle\psi_-\right| + \left|\psi_+\right\rangle\left\langle\psi_+\right|\right) \\ &= \left(\frac{3x+1}{4}\right)\left|\psi_-\right\rangle\left\langle\psi_-\right| + \frac{1-x}{4}\underbrace{\left(\left|\psi_+\right\rangle\left\langle\psi_+\right| + \left|00\right\rangle\left\langle 00\right| + \left|11\right\rangle\left\langle 11\right|\right)}_{\text{triplet}}\end{aligned} \tag{8.42}$$

therefore $\hat{\rho}(1) = \left|\psi_-\right\rangle\left\langle\psi_-\right|$, $\hat{\rho}(0) = I/4$.

The state

$$\hat{\rho}(x) = \sum_{i=1}^{4} w_i(x)\left|\phi_i\right\rangle\left\langle\phi_i\right| ,$$

where $w_i(x) \in [0,1]$, does not contain a clue about the location of a geometric border that separates the region of separable qubits states from the entangled.

The parameter x can be also interpreted as the intrinsic time evolution of the system which was initially prepared in the pure singlet state $\left|\psi_-\right\rangle\left\langle\psi_-\right|$ and, by interacting with a specific environment, it evolves and finally attain, asymptotically, the stochastic state I. Assuming that T is a characteristic time of the system, one possibility is to relate x to time t as $x(t) = \exp(-t/T)$, thus $x(0) = 1$ and $\lim_{t\to\infty} x(t) = 0$.

In Fano's representation, BEW state has no polarization as $P_{1,z} = P_{2,z} = 0$, it only has non-null diagonal entries in the correlation matrix, that are degenerated, $M_{xx} = M_{yy} = M_{zz} = -x$, thence the former seven possible "free parameters" are reduced to only one

$$\left(\begin{array}{ccccccc} P_{1,z} & P_{2,z} & M_{xx} & M_{yy} & M_{zz} & M_{xy} & M_{yx} \end{array}\right) = \left(\begin{array}{ccccccc} 0 & 0 & -x & -x & -x & 0 & 0 \end{array}\right). \tag{8.43}$$

In matrix form, the 2-qubit system state (8.35) is

$$\hat{\rho}(x) = \frac{1-x}{4} I \otimes I + \frac{x}{4}\left(\sigma_z \otimes \bar{\sigma}_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y\right) \tag{8.44a}$$

$$= \frac{1}{4}\left(I \otimes I + x\sigma_z \otimes \bar{\sigma}_z\right) - \frac{x}{4}\left(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y\right) \tag{8.44b}$$

$$= \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & 1+x & -2x & 0 \\ 0 & -2x & 1+x & 0 \\ 0 & 0 & 0 & 1-x \end{pmatrix} \tag{8.44c}$$

the σ's are the Pauli matrices[1] and $\bar{\sigma}_z = \mathrm{diag}\,(-1,1)$ where $\mathrm{Tr}\hat{\rho}^2\,(x) = \left(3x^2+1\right)/4 \leq 1$. It is a mixed state for $x<1$ and pure only at $x=1$. The eigenstates and eigenvalues are

$$\left|\psi_-\right\rangle \leftrightarrow \lambda_s\,(x) = \frac{1}{4}\,(3x+1)\,, \tag{8.45}$$

for the singlet, and

$$\left\{|11\rangle\,,\ |00\rangle\,,\left|\psi_+\right\rangle\right\} \leftrightarrow \lambda_t\,(x) = \frac{1}{4}\,(1-x)\,, \tag{8.46}$$

for the triplet, displaying a triple degeneracy. For $x=1/3$, $\lambda_s\,(1/3)=1/2$ and $\lambda_t\,(1/3) = 1/6$. The sets of modified parameters are

$$(t_-, u_-, v_+, w_-) = \left(\frac{1+x}{2}, 0, -x, 0\right) \tag{8.47a}$$

$$(t_+, u_+, v_-, w_+) = \left(\frac{1-x}{2}, 0, 0, 0\right) \tag{8.47b}$$

8.5.1 ***Quadridistance*** and ***quadrispeed*** in $4D$

The two quadratic *quadridistances* are

$$s_1^2\,(x) = t_-^2 - \left(u_-^2 + v_+^2 + w_-^2\right) = t_-^2 - X_1^2 = \frac{1}{4}\,(1-x)\,(1+3x) \geq 0 \tag{8.48a}$$

$$s_2^2\,(x) = t_+^2 - u_+^2 - v_-^2 - w_+^2 = t_+^2 - X_2^2 = \left(\frac{1-x}{2}\right)^2 - 0 \geq 0, \tag{8.48b}$$

the distances between two "events" in 3D are $X_1^2 = (-x)^2$ and $X_2^2 = 0$. The sum of the quadridistances and their ratio are

$$s_1^2\,(x) + s_2^2\,(x) = \frac{1}{2}\left(1-x^2\right) \leq \frac{1}{2} \tag{8.49a}$$

$$\frac{s_2^2\,(x)}{s_1^2\,(x)} = \frac{1-x}{1+3x} \leq 1, \tag{8.49b}$$

although $s_1^2\,(0) = s_2^2\,(0) = 1/4$, $s_1^2\,(x)$ and $s_2^2\,(x)$ differ from each other at $x=1$, $s_1^2\,(1)=1$ and $s_2^2\,(1)=0$. For a pure state the quadridistance $s_2\,(1)$ is null whereas $s_1\,(1)$ attains its maximum value. When the state is completely stochastic both quadridistances acquire the same value, $s_1\,(0) = s_2\,(0) = 1/2$. One can find the light cone locations solving the equation

$$t_-^2\,(x) - X_1^2\,(x) = 0 \Longrightarrow \frac{1}{4}\,(3x+1)\,(1-x) = 0 \tag{8.50}$$

the two zeros of this equation lead to

$$x_{11} = 1 \Longrightarrow \begin{pmatrix} t_-^2\,(1) & X_1^2\,(1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$x_{12} = -1/3 \Longrightarrow \begin{pmatrix} t_-^2\,(-1/3) & X_1^2\,(-1/3) \end{pmatrix} = \begin{pmatrix} 1/9 & 1/9 \end{pmatrix}$$

[1] Worth noting that the overline was introduced to define $\bar{\sigma}_z = -\sigma_z$ because $\sigma_z \otimes (-\sigma_z) \neq -\left(\sigma_z \otimes \sigma_z\right)$.

and $M_{xx}(x_{12}) = M_{yy}(x_{12}) = M_{zz}(x_{12}) = -x_{12} = 1/3$. The partner of eq. (8.50) is

$$\begin{aligned} t_+^2(x) - X_2^2(x) &= 0 \Longrightarrow \left(\frac{1-x}{2}\right)^2 = 0 \Longrightarrow x_{21} = 1, \\ &\Longrightarrow \begin{pmatrix} t_+^2(1) & X_2^2(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \end{aligned}$$

The plots of parametric components

$$\begin{aligned} \begin{pmatrix} s_1^2(x) & t_-^2(x) \end{pmatrix} &= \begin{pmatrix} \frac{1}{4}(1-x)(1+3x) & \left(\frac{1+x}{2}\right)^2 \end{pmatrix}, \quad (8.51a) \\ \begin{pmatrix} s_2^2(x) & t_+^2(x) \end{pmatrix} &= \begin{pmatrix} \left(\frac{1-x}{2}\right)^2 & \left(\frac{1-x}{2}\right)^2 \end{pmatrix}, \quad (8.51b) \end{aligned}$$

are presented in figs. 8.1 and 8.2.

For the partially transposed matrix,

$$\hat{\rho}^T = \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & -2x \\ 0 & 1+x & 0 & 0 \\ 0 & 0 & 1+x & 0 \\ -2x & 0 & 0 & 1-x \end{pmatrix}, \quad (8.52)$$

to be compared with state (8.44c), as the parameters are

$$\begin{aligned} (t_-, 0, v_-, 0) &= ((1+x)/2, 0, 0, 0), \quad (8.53a) \\ (t_+, 0, v_+, 0) &= ((1-x)/2, 0, -x, 0), \quad (8.53b) \end{aligned}$$

therefore, the quadratic quadridistances become

$$\begin{aligned} \left(s_1^T\right)^2 &= t_-^2 - v_-^2 = \left(\frac{1+x}{2}\right)^2, \quad (8.54a) \\ \left(s_2^T\right)^2 &= t_+^2 - v_+^2 = \frac{1}{4}(x+1)(1-3x), \quad (8.54b) \end{aligned}$$

so $\left(s_2^T\right)^2$ assumes negative values for $x \in (1/3, 1]$, meaning that according to $\hat{\rho}^T(x)$ the qubits are entangled for x in this interval. Numerically, $\left(s_1^T\right)^2\Big|_{x=0} = \left(s_2^T\right)^2\Big|_{x=0} = 1/4$, whereas $\left(s_1^T\right)^2\Big|_{x=1} = 1$ and $\left(s_2^T\right)^2\Big|_{x=1} = -1$. Thence

$$1/4 \leq \left(s_1^T\right)^2\Big|_{x\in[0,1]} \leq 1 \quad (8.55)$$

and

$$-1 \leq \left(s_2^T\right)^2\Big|_{x\in[1,0]} \leq 1/4. \quad (8.56)$$

The range for $\left(s_2^T\right)^2$ is greater than for $\left(s_1^T\right)^2$, see Fig 8.1 One may also admit that the system is evolving in time, t being an intrinsic parameter; we may assume, for instance, that $x(t) = \exp(-\gamma t)$, so $t_- = (1+\exp(-\gamma t))/2$, $t_+ = (1-\exp(-\gamma t))/2$ and $v_+ =$

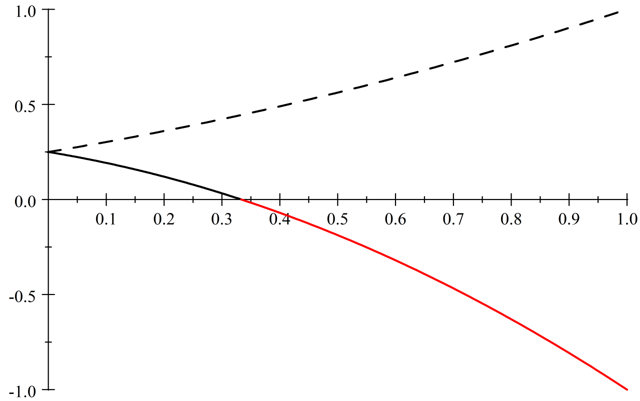

Figure 8.1: The plot of $\left(s_1^T\right)^2$ is the dashed line, no significant information. For $\left(s_2^T\right)^2$, the solid line in black stands for the separable state, in red color it represents the x values for the entangled state.

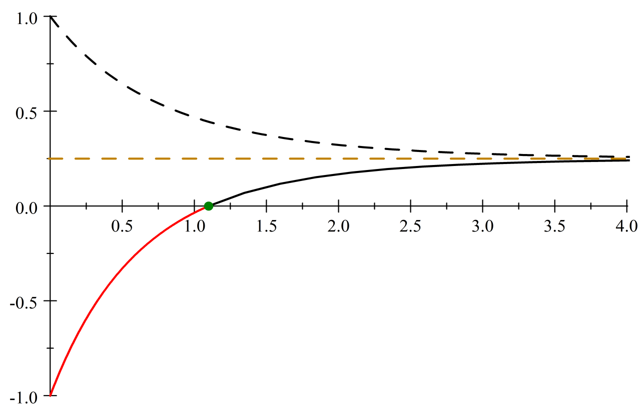

Figure 8.2: *Evolution in time*: The black dashed curve is for $\left(s_1^T\left(t\right)\right)^2$ vs t. For $\left(s_2^T\left(t\right)\right)^2$ vs t, the solid line in red color is for the entangled state; the solid black line is for the separable qubits state. The dot in green, on the t axis stands for $\ln 3$. The dashed line in siena color marks the point of the asymptotic stochastic state.

$-\exp\left(-\gamma t\right)$. At $t = 0$ the qubits are in a singlet state while at $t \longrightarrow \infty$ the state becomes maximally mixed.

Concerning the speeds, from eq. (8.53a)

$$V_1^T = \frac{dv_-}{dt_-} = \frac{dv_-/dx}{dt_-/dx} = 0 \tag{8.57}$$

and from (8.53b)

$$V_2^T = \frac{dv_+}{dt_+} = \frac{dv_+/dx}{dt_+/dx} = 2 \tag{8.58}$$

the quadrispeeds become

$$\left(\frac{ds_1^T}{dt_-}\right)^2 = 1 - \left(V_1^T\right)^2 = 1 \tag{8.59a}$$

$$\left(\frac{ds_2^T}{dt_+}\right)^2 = 1 - \left(V_2^T\right)^2 = 1 - 4 = -3. \tag{8.59b}$$

while the first speed evolves on the "light cone", the second is "superluminal". Both speeds, are independent of x as they dependence on x is linear; for other quantum states [12–15], having a multiparametric nonlinear dependence, the quadrispeeds may depend on further parameters.

8.6 Summary and conclusion

I studied a common problem in quantum mechanics involving the relationship between entanglement and separability for a two-qubit system. The state of the system, in Fano's form, is described using two polarization vectors and a correlation matrix. By examining the symmetries contained in the state, represented as a 4×4 matrix, I introduced new parameters by reordering the matrix entries. These parameters show that the positivity condition of the state is related to quadratic distances in 4D (3+1) space, under a Minkowski metric on a compact space.

By performing a local reflection symmetry operation on the bipartite state one obtains another one, which is equivalent to that obtained by doing a matrix partial transposition as proposed in [8, 9]). This procedure permits the construction of two quadridistances, and for certain values of the parameters one can be negative, therefore, according to the Peres-Horodecki criterion (PHC) the state cannot be expressed in a separable form, the qubits are objectively entangled.

In Minkowski's compact space, the PHC has a geometrical interpretation. One can draw trajectories and visually identify two regions: one where the qubits are entangled and another where they are separable. The border between these regions is equivalent to the light cone surface in SR. Using a lexicon adapted from the SR, one region has been called entangled-like, equivalent to space-like, and the other separable-like, analogous to time-like.

It is surprising to find a connection between quantum entanglement and the Minkowski structure of special relativity, despite the lack of an obvious link between both. The formalism and example presented in this essay, as well as previous research [6, 7, 22], demonstrate a parallelism between the two fields of physics. It is worth noting that the speed of disentanglement corresponds to the speed of a superluminal object in SR.

Acknowledgments

Partial support from National Council for Scientific and Technological Development – CNPq (Brazil) is acknowledged.

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Cite this work as:

S. S. Mizrahi, Speed of disentanglement for a two-qubit system staged in a Minkowski space with compact support, in A. Dodonov and C. C. H. Ribeiro (Eds.), Proceedings of the Second International Workshop on Quantum Nonstationary Systems, pp. 123–138 (LF Editorial, São Paulo, 2024). ISBN: 978-65-5563-446-4.

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