KNOTS AND QUANTUM MECHANICS (III)

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Plan of the Third Lecture

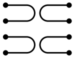
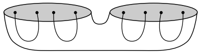
1. Entanglement classification
2. Entropy inequalities
3. Quantum teleportation

Entanglement Classification

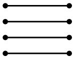
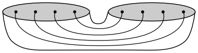
Preliminary classification

We already saw that topology distinguishes different types of entanglement:

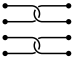
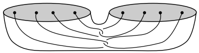
- separable states

 $\sim$  $= |0\rangle|0\rangle$

- Maximally entangled states

 or  $\sim$  $= |0\rangle|0\rangle + |1\rangle|1\rangle$

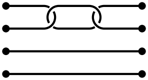
- Less entangled states (non-unitary matrices)

 $\sim$ 

Entanglement Classification

Classification

There is a discrete infinite number of topologies that can represent different types of entanglement



- **Conceptual Problem 1:** In Quantum Resource Theory one is more interested in a finite classification: States from different classes are suitable for different quantum tasks
- **Conceptual Problem 2:** Can one produce a complete classification?
- **Technical Problem:** For finite integer k not all the diagrams are independent, some topologies are equivalent

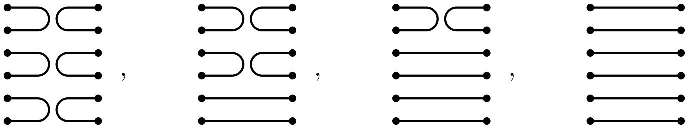
What is the best way to coarse-grain the topological classification?

Entanglement Classification

Connectivity

- A simple way to characterize wiring is through connectivity, that is by telling what is connected to what
- The classes of connectivity can be defined by corresponding (adjacency matrices of) graphs

In the case of bipartition the simplest representatives of the connectivity classes are planar graphs

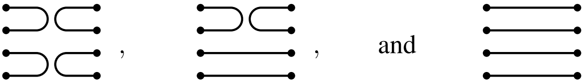


All the planar diagrams modulo local braiding

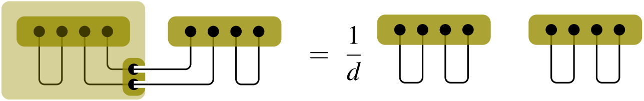
Entanglement Classification

Pair of qubits

For a pair of S^2 with four punctures the only planar diagrams modulo local action are



- The first state is separable and the last one is maximally entangled
- The middle state is the same as the first one (two lines not enough to support entanglement)

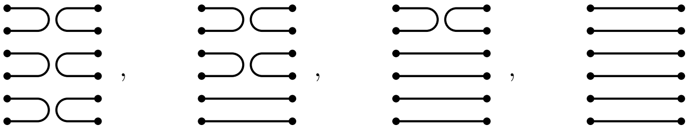


There are only two nonequivalent classes of diagrams

Entanglement Classification

Bipartite entanglement beyond qubits

A pair of spheres with six punctures have dimension $C_3 = 5$



But there are only three independent planar connectomes

- This generalization is not easily extendable to arbitrary qudits
- How can we generalize systematically?

Entanglement Classification

Jones-Wenzl projectors

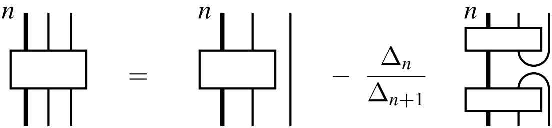
The space of spin j representation has dimension $2j + 1$

- There are $2j + 1$ ways of making a singlet out of 4 spins j.

It will be convenient to construct spin j as $2j$ spins $1/2$. Note that


is a projector on a singlet

For spin $j = n/2 + 1$ one can use (thick line = n normal ones)

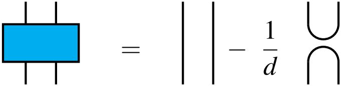


$$\Delta_{-1} = 0\,, \qquad \Delta_0 = 1\,, \qquad \Delta_{n+1} = d\Delta_n - \Delta_{n-1}$$

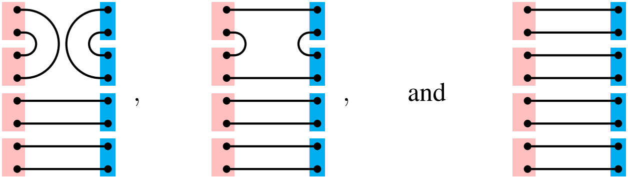
Entanglement Classification

Qutrit entanglement

Use Jones-Wenzl projectors joining pairs of lines into spin $j = 1$



Find all independent diagrams modulo local braiding:



- Diagrams with more broken connections are equivalent to the first one
- For classification it is enough to consider only half of the punctures

Entanglement Classification

General bipartite entanglement

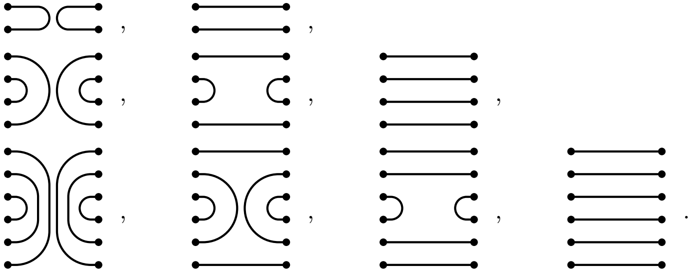
One can generalize the diagrammatic classification to $\mathcal{H} = \mathbb{C}_n \otimes \mathbb{C}_m$

- For $n = m$ the set of diagrams is ($2n - 2$ points for $\mathbb{C}^n$)

$n = 2$:

$n = 3$:

$n = 4$:



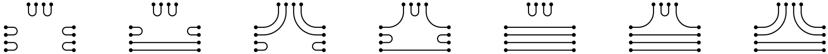
- For $n \neq m$ one adds points to one of the sides

Entanglement Classification

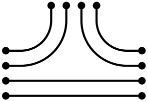
Tripartition

For the bipartite case the simple classification gives a very intuitive picture of entanglement

- For the tripartite case (qubits) the same classification should give



There are only 3 nonequivalent classes. The genuine tripartite entanglement



$$= |000\rangle + \frac{1}{\sqrt{d^2-1}}|111\rangle, \qquad d = -A^2 - A^{-2}$$

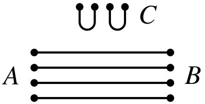
is of GHZ type. No obvious W state (e.g. $|001\rangle + |010\rangle + |100\rangle$)

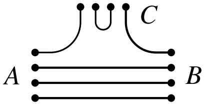
Properties of Entanglement

Monogamy

- Entanglement is a resource that can be shared between different systems
- In the TQFT approach the resource are the Wilson lines

Monogamy of entanglement means that a state that is maximally entangled with a second state cannot be entangled with any other state



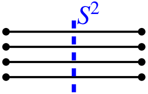


Properties of Entanglement

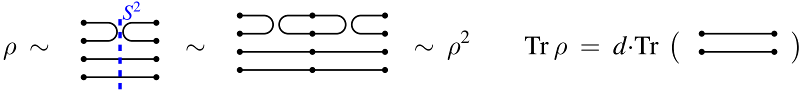
Entropy of the connectome states

Let us focus on the simple class of projector states

- For the maximally entangled state the entropy is given by the dimension of the Hilbert space



- This Hilbert space is that of a sphere that dissects the state
- One can repeat the replica trick derivation for other connectomes



- The entropy is given by S^2 cutting the minimum number of lines

Properties of Entanglement

Line counting

- For the connectome states the entropy is given by $\log(\dim \mathcal{H}_{\rm min})$
- $\mathcal{H}_{min}$ is the Hilbert space of S^2 with the minimal number of punctures, cutting the state into two
- For $k \gg n \gg 1$

$$\dim \mathcal{H}_{2n} = \frac{(2n)!}{(n+1)!n!} \simeq \frac{4^n}{n^{3/2}\sqrt{\pi}}\,, \qquad S_{\rm E} \simeq 2n \log 2$$

$\log 2$ of entropy per line

Instead of entropy one can measure entanglement by the number of lines

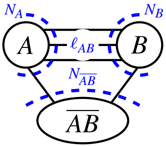
Properties of Entanglement

Subadditivity

Let us prove some of the characteristic equalities by line counting

- Subadditivity
$$S(AB) \;\leq\; S(A) + S(B)$$
Denote
  - $\overline{AB}$ – the complement of AB
  - N_A and N_B – the numbers of lines that emanate from A and B
  - ℓ_{AB} – the number of lines connecting A and B
  - $N_{\overline{AB}}$ is the number of lines connecting both A and B to $\overline{AB}$

Then
$$N_A + N_B \;=\; N_{\overline{AB}} + 2\ell_{AB} \;\geq\; N_{\overline{AB}}$$



ℓ_{AB} is mutual information $I(AB) = S(A) + S(B) - S(AB)$

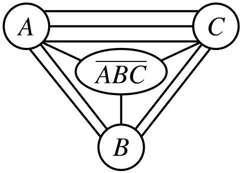
Properties of Entanglement

Strong subadditivity

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$$

Denote

- $N_{\overline{ABC}}$ – total number of lines that connect A, B or C with $\overline{ABC}$
- $N_{\overline{AB}}$ and $N_{\overline{BC}}$ – number of lines connecting the unions with their complements
- N_B – number of lines emanating from B
- ℓ_{AC} – number of lines connecting A and C



Then

$$N_{\overline{ABC}} \;=\; N_{\overline{AB}} + N_{\overline{BC}} - 2\ell_{AC} - N_B \leq N_{\overline{AB}} + N_{\overline{BC}} - N_B$$

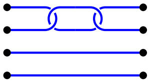
Emergence of Spacetime

Classical limit

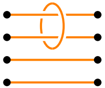
In Chern-Simons $k \to \infty$ corresponds to the classical limit

$A \to 1$,

Braiding $\to$ permutations (no knots). All states are simple connectomes



$\to |00\rangle$



$\to |00\rangle + |11\rangle$

Large k behavior of the entropy (minimal area and corrections)

$$S = \frac{\pi^4}{k^4}\left(1 - \log\frac{\pi^4}{k^4}\right) + O(k^{-5})$$

$$S = \log 2 - \frac{8\pi^4}{k^4} + O(k^{-5})$$

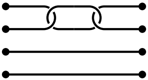
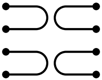
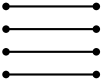
Emergence of Spacetime

Classical and quantum topologies

Connectome states are similar to classical geometries in holography

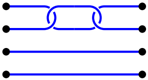
- We can label them *classical topologies*

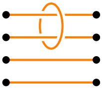
Quantum topologies are linear combinations of the classical ones:

 $= (A^4+A^{-4}-1)$  $+(1-A^{-4})(1-A^4)$ 

Defects contribute to the relative weight of the disconnected topologies

Emergence of space?

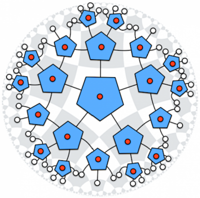




Connection to (Loop) Quantum Gravity

Emergence of Spacetime

AdS/CFT



- Quantum gravity (string theory) in AdS space is equivalent to a CFT at the boundary of AdS

This equivalence can be understood in the sense of state-cobordism correspondence:

- Topology of the space glued to the boundary encodes a quantum state

The analogy can be made even more precise if one recalls that gravity in AdS_3 can be cast as $SL(2,R) \times SL(2,R)$ Chern-Simons theory

Classical gravity limit:

$$G_N \to 0\,, \qquad \frac{L^{d-2}}{G_N} \gg 1$$

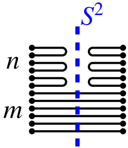
Emergence of Spacetime

Ryu-Takayanagi formula

For two regions A and B on the AdS boundary the entanglement entropy is

$$S_A = S_B = \min_{\gamma_{AB}} \frac{\text{Area}[\gamma_{AB}]}{4G_\text{N}}$$

Recall that for the projector states the entropy is defined by the number of connections:

$|\Psi\rangle =$ 

$$S_A = S_B = m \log 2\,.$$

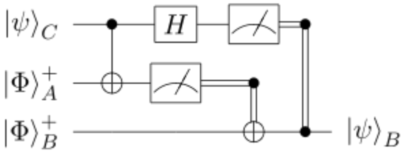
valid for $k \gg m \gg 1$

- One can assume that each line is a flux with area $4G_N \log 2$

Quantum Protocols

Quantum teleportation [Bennett et al.'93]

Alice has an unspecified quantum state which she wants to pass to Bob. They share an entangled pair:

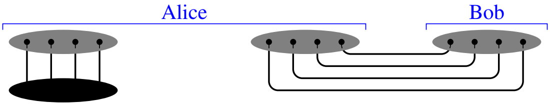


- Alice applies an entangling transformation on her pair of states
- She measures the result an reports it to Bob via a classical channel
- Depending on the result of the measurement Bob recovers the original state by an appropriate unitary

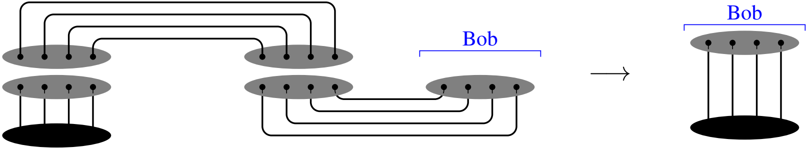
Quantum Protocols

Teleportation in TQFT

Consider the teleportation setup in a topological context



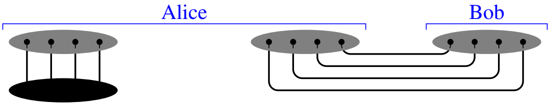
Teleporting the state to Bob means



Quantum Protocols

Teleportation in TQFT [Coecke][Kauffman][DM]

Consider the teleportation setup in a topological context



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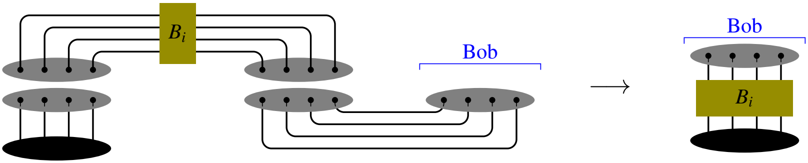
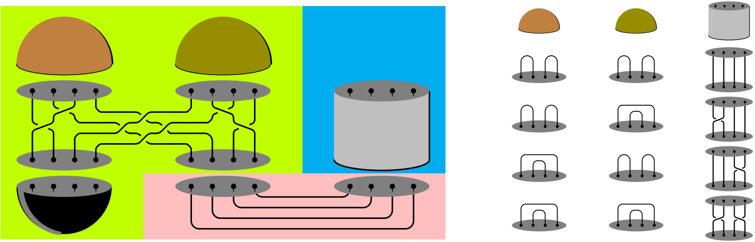


Quantum Protocols

Classical version

Principle of the quantum algorithm can be illustrated by a classical diagram



This is not a proper quantum algorithm

- Here projectors are involved (stochastic algorithm)
- The projector basis is not orthogonal

Conclusions

- TQFT offers a way to visualize quantum mechanics in terms of space diagrams
- Knot theory provides calculational tools for this presentation of quantum mechanics
- We mainly focused on quantum entanglement and showed how some of its properties can be illustrated by the topological approach
- Perhaps such approach can be applied to quantum gravity and quantum computation (at least for educational purposes)