KNOTS AND QUANTUM MECHANICS (II)

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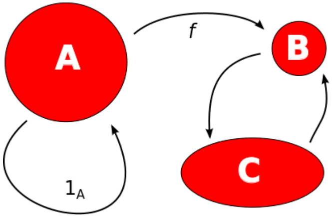
Plan of the Second Lecture

1. TQFT
2. Topological quantum mechanics
3. Quantum entanglement

Categorification

Basics of category theory

Category theory is a general theory of mathematical structures and their relations.



Each category consists of

- objects $A, B, \ldots$
- morphisms $f : A \longrightarrow B, \ldots$

Axioms

- for $f : A \to B$ and $g : B \to C$ $\exists$ *composition* $g \circ f : A \to C$
- $\forall A\ \exists!\ 1_A : A \to A$ such that $f \circ 1_A = f$ or $1_A \circ g = g$
- composition is associative $h \circ (g \circ f) = (h \circ g) \circ f$

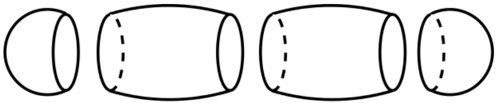
Fundamental example: *Vect* – linear spaces and linear maps

Categorification

$Cob[d]$ – cobordism category in d dimensions

- objects – codimension-one oriented surfaces Σ in d dimensions
- morphisms $\mathcal{M} : \Sigma_1 \to \Sigma_2$ – dimension d manifolds with boundary $\partial M = \Sigma_2 \cup \overline{\Sigma}_1$, where $\overline{\Sigma}$ is Σ with inverted orientation

Composition



Identity 1_Σ is a "cylinder" $\Sigma \times I$

Categorification

Functors – maps between categories

Each functor F includes

- identification of objects $F(A) = \mathcal{A}$
- identification of morphisms $F(g) \ = \ \mathcal{G}$

Axioms (F preserves the structure of the category)

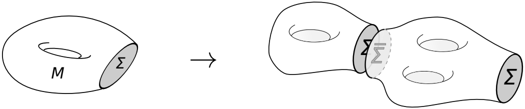
- $F(1_A) = 1_{F(A)}$
- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(g \circ f) = F(g) \circ F(f)$

Topological Quantum Field Theory (TQFT) in d dimensions is a functor between the category of cobordisms $Cob[d]$ and the category of vector spaces $Vect$

Categorification

Formal definition [Atiyah'89]

- Functor Z from $Cob[d]$ to $Vect$:
  1. $(d-1)$-dimensional $\Sigma \longrightarrow$ vector space $\mathcal{H}_\Sigma = Z(\Sigma)$
  2. d dimensional M, $\Sigma = \partial M \longrightarrow$ vector $|\psi\rangle = Z(M) \in \mathcal{H}_\Sigma$
  3. $\forall\ \Sigma_1, \Sigma_2$ and M, $\partial M = \overline{\Sigma}_1 \cup \Sigma_2$, $\longrightarrow$ linear map $U = Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$

$|\psi\rangle =$  $= U|\psi\rangle$

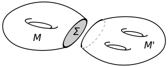
H_Σ are Hilbert spaces and states $|\Psi\rangle$ in these Hilbert spaces are encoded by different d-manifolds glued to Σ

Categorification

Atiyah's axioms

- M with no boundary is a state in a trivial Hilbert space (single point).
  $M \;\to \mathbb{C}$-number

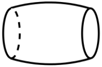
  In particular, there is an obvious scalar product



$\longrightarrow$

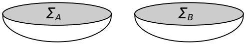
$\equiv \; \langle \psi | \psi' \rangle$

- There is an identity operator, which corresponds to a featureless cylinder connecting a pair of Σ



$\to \; \mathbb{I}$

- $Z(\Sigma_1 \cup \Sigma_2) \;=\; Z(\Sigma_1) \otimes Z(\Sigma_2)$ for a disjoint union



$\longrightarrow \; \mathcal{H}_{\Sigma_A} \otimes \mathcal{H}_{\Sigma_B}$

Chern-Simons Theory

Explicit map

- The previous construction is a definition of a *functor* Z relating *categories* of topological and linear spaces
- One explicit realization of such a functor is given by partition functions of Chern-Simons theories (path integrals)

$$S_{\mathrm{CS}}[\mathcal{M}] \;=\; \frac{k}{4\pi}\int_{\mathcal{M}} d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right)$$

Consider 3D space $\mathcal{M}$ with boundary Σ. Then a state is

$$\Psi(\Sigma) \;=\; \int \mathcal{D}A\Big|_{A(\Sigma)=A_\Sigma} \mathrm{e}^{iS_{\mathrm{CS}}[\mathcal{M}]}$$

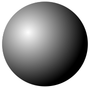
Scalar product (composition)

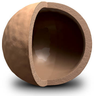
$$\langle\Phi|\Psi\rangle \;=\; \int \mathcal{D}A_\Sigma \int \mathcal{D}A\Big|_{A(\Sigma)=A_\Sigma} \int \mathcal{D}\bar{A}\Big|_{\bar{A}(\Sigma)=A_\Sigma} \mathrm{e}^{iS_{\mathrm{CS}}[\mathcal{M}_1\cup\mathcal{M}_2]}$$

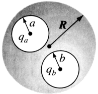
Chern-Simons Theory

Specific model

We will be working with $SU(2)$ Chern-Simons theory on 3-manifolds with multiple $\Sigma = S^2$ boundaries. Hilbert space $\mathcal{H} = \mathcal{H}_{S^2} \otimes \mathcal{H}_{S^2} \otimes \cdots$

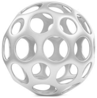


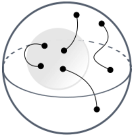




It turns out $\dim \mathcal{H}_{S^2} = 1$

- Need to make holes ($\Sigma = S^2 \backslash \{P_i\}$)
- Minimum working example – S^2 with 4 punctures – qubit





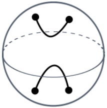
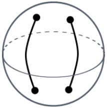
Chern-Simons Theory

Qubit

Punctures are non-dynamical particles characterized by irreps J of $su(2)_k$ in WZW theory

- A non-trivial $\mathcal{H}_n$ exists if $\otimes_i^n J_i$ contains trivial irreps
- $\dim \mathcal{H}_n$ is the number of trivial irreps

For 4 particles in spin 1/2 irrep $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1}$, so

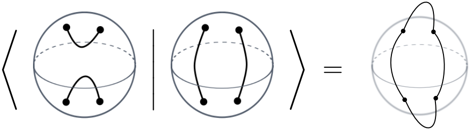
$\dim \mathcal{H} = 2$ $\quad |0\rangle \sim$  $\quad |1\rangle \sim$  $- \frac{1}{N} |0\rangle$

Irreps of $su(2)_k$ are a bit different from irreps of $su(2)$ – some irreps might be trivial

Chern-Simons Theory

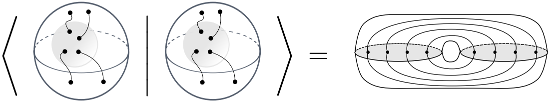
Scalar products

To compute an overlap of two qubit states one has to glue two 3-balls along the boundary 2-sphere:



Expectation value of the Wilson loop operator in S^3 (Jones polynomial)

Multiqubits



$= Z(S^2 \times S^1; R_1, R_2, R_3, R_4)$

The resulting space is a higher order topology ($S^2 \times S^1$)

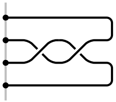
Topological Quantum Mechanics

1D TQFT

Let's try to construct a 1D example

Σ are just collections of points (on a line)

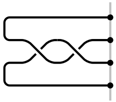
What are the vectors in $\mathcal{H}$?

$|\psi\rangle =$ 

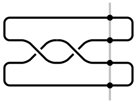
Vectors in $\mathcal{H}$ are all ways of connecting points (on the right of the line) and their linear combinations

- How does one compute inner product in this space?

bra: Flip the diagram (exchange undercrossings and overcrossings)

 $= \langle\psi|$

Glue bra and ket!

$\langle\psi|\chi\rangle =$ 

The result has no open ends, so it is not a vector, but a number

Topological Quantum Mechanics

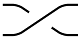
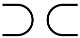
Basis

- What is the dimension of $\mathcal{H}$?

Fix it by requiring that $\mathcal{H}$ can be spanned by diagrams that connect points without intersection (Temperley-Lieb basis). For example, for 4 points,

$|e_0\rangle =$  , $|e_1\rangle =$ 

For this to make sense we need linear relations for the intersections. We will require that

 $= A$  $+ A^{-1}$ 

- Note that the number of points must be even
- For $2n$ points the number of the basis diagrams is

$$\dim \mathcal{H}_{2n} = \frac{(2n)!}{(n+1)!(n)!} = 1, 1, 2, 5, 14, \ldots \quad \text{(Catalan numbers)}$$

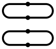
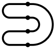
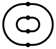
Topological Quantum Mechanics

Calculus

Temperley-Lieb basis is not orthonormal. Use Gram-Schmidt procedure

$$|0\rangle \equiv \frac{1}{\sqrt{\langle e_0|e_0\rangle}}$$  ,
$$|1\rangle \equiv \frac{1}{N}\Bigg($$  $$- \frac{\langle e_0|e_1\rangle}{\langle e_0|e_0\rangle}$$  $$\Bigg)$$

How does one compute the overlaps?

$$\langle e_0|e_0\rangle =$$  ,
$$\langle e_0|e_1\rangle =$$  ,
$$\langle e_1|e_1\rangle =$$ 

Require that these are given by the 'Jones' polynomials of the diagrams

- Jones polynomial of a trivial circle $J(\bigcirc) = d$
- For any link $\mathcal{L}$, $J(\bigcirc \cup \mathcal{L}) = d \cdot J(\mathcal{L})$

For the rest use skein relations. Consistency condition: $d = -A^2 - A^{-2}$

Topological Quantum Mechanics

Further comments

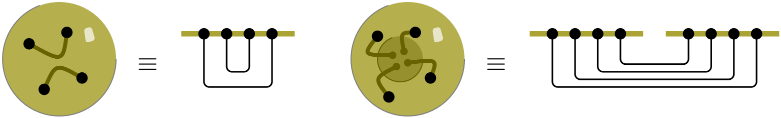
Dimension revisited: Compute the Gram matrix

$$\det\langle e_i|e_j\rangle \;=\; d^2(d^2-1)$$

The dimension is 2 unless $d = 0, 1$. In general

$$\dim \mathcal{H}_{2n} \;=\; C_n\,, \quad \text{if} \quad k > n-1 \;\left(d = -2\cos\frac{\pi}{k+2}\right)$$

This 1D TQFT is equivalent to $SU(2)_k$ Chern-Simons theory



Quantum Entanglement

In ordinary quantum mechanics

Consider a multipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$, state $|\Psi\rangle \in \mathcal{H}$ and its density matrix

$$\rho = |\Psi\rangle \otimes \langle\Psi|$$

Let us define the *reduced density matrix* for subsystem $\mathcal{H}_1$

$$\rho_1 = \text{Tr}_{\mathcal{H}_2, \mathcal{H}_3, \ldots} (\rho)$$

- $\mathcal{H}_1$ is not entangled with (separable from) the rest of $\mathcal{H}$ if $\rho_1 = |\Psi_1\rangle \otimes \langle\Psi_1|$ for some $|\Psi_1\rangle$
- otherwise, $\mathcal{H}_1$ is entangled

Example: in the EPR (Bell) state one has a pair of entangled spins

$$|\Psi_{12}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \right)$$

Quantum Entanglement

Von Neumann (entanglement) entropy

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B : \quad S = -\mathrm{Tr}_A(\rho_A \log \rho_A), \qquad \rho_A = \mathrm{Tr}_{\bar{A}}(\rho)$$

von Neumann entropy is a measure of entanglement

Replica trick: Compute instead (Rényi entropies)

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}_A\left(\rho^n\right)$$

Analytically continue in n and find S in the limit $n \to 1$

Example

EPR: $$\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad S_n = \frac{1}{1-n} \log\left(2\frac{1}{2^n}\right) = \log 2$$

Quantum Entanglement

SLOCC classification

States with different entanglement properties are suitable for different quantum tasks. General problem: *classify different types of entanglement*

- There is a partial classification known as **SLOCC** (*Stochastic Local Operations and Classical Communication*)

For the local separation $\mathcal{H} = \mathcal{H}_{N_1} \otimes \cdots \otimes \mathcal{H}_{N_n}$ SLOCC classes are orbits of the action of $SL(N_1) \otimes \cdots \otimes SL(N_n)$:

$$\frac{\mathbb{C}^{N_1} \otimes \cdots \otimes \mathbb{C}^{N_n}}{SL(N_1) \otimes \cdots \otimes SL(N_n)}$$

- For 2 qubits – two SLOCC classes: separable and entangled (Bell)
- For two-partite $\mathcal{H} = \mathbb{C}^m \times \mathbb{C}^n$, with $m \geq n$, there are n classes
- For three qubits there four classes: separable, Bell, GHZ and W
- For four qubits or three qutrits: no finite classification

Quantum Entanglement

Two-partite systems

Consider the situation $\mathcal{H} = \mathbb{C}^m \times \mathbb{C}^n$, with $m \geq n$

- The reduced density matrix ρ_m or ρ_n can have at most rank n

$$\text{Schmidt:} \quad \rho \;=\; \sum_{i=1}^{n} \lambda_i |i\rangle\langle i|\,, \quad \lambda_i \geq 0$$

The SLOCC class is determined by the number of $\lambda_i \neq 0$

- Local unitary operations do not change the von Neumann entropy (preserve entanglement)
- Local invertible operations will in general decrease the entropy

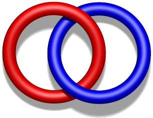
General result: Local operations can convert states with more entanglement into states with less entanglement with certainty, but only probabilistically in the opposite direction

Quantum Entanglement and Knots

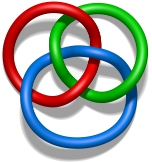
Entanglement: quantum vs topological

It seems quite natural to interpret entanglement as a sort of knotting or linking. In 1997 Aravind suggests the following:

- EPR state: $\psi = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$



- GHZ state: $\psi = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)$



Here a circle corresponds to a spin, while linking of circles is entanglement of spins

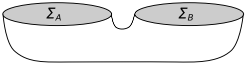
Quantum Entanglement and Topology

What does entanglement mean in terms of topology?

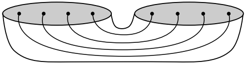
The bra-ket notation is highly suggestive: separable $\rho = |\Psi\rangle\langle\Psi|$

Consider $\Sigma = \Sigma_A \cup \Sigma_B$. Two classes of states (3D topologies):

$|\Psi_1\rangle =$ 

$|\Psi_2\rangle =$ 

- $|\Psi_1\rangle$ is always separable
- Separability of $|\Psi_2\rangle$ depends on $\dim \mathcal{H}$



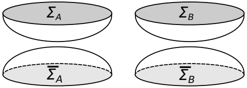
Entanglement = space? Space emerges from entanglement? ER=EPR?

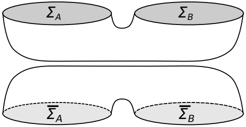
Quantum Entanglement and Topology

Replica trick [Dong,Fradkin,Leigh,Nowling'08]

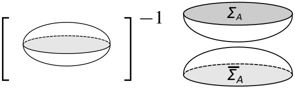
- compute $\mathrm{Tr}\,\rho_A^n$ $\qquad S = -\frac{d}{dn}\,\mathrm{Tr}\,\rho_A^n\Big|_{n=1}$

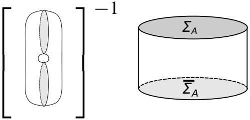
(Unnormalized) density matrices for states $|\Psi_1\rangle$ and $|\Psi_2\rangle$

$\widehat{\rho}_1 =$ 

$\widehat{\rho}_2 =$ 

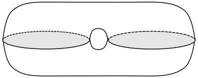
Normalized reduced density matrices

$\rho_1(A) =$ 

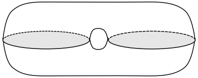
$\rho_2(A) =$ 

Quantum Entanglement and Topology

Entanglement entropy

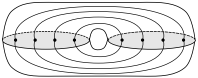
$$\mathrm{Tr}\,\left(\rho_1^A\right)^n = 1\,, \qquad \mathrm{Tr}\,\left(\rho_2^A\right)^n = \Bigg[$$

$$\Bigg]^{1-n}$$

Consequently,

$$S_\mathrm{E}(\rho_1) = 0\,, \qquad S_\mathrm{E}(\rho_2) = \log\Bigg[$$

$$\Bigg]$$

The entropy of $|\Psi_2\rangle$ is computed by a topological invariant on $\Sigma \times S^1$. The value depend on the features of topology

- One can guess that

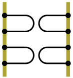
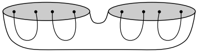
$$\log\Bigg[$$

$$\Bigg] = \log \mathrm{Tr}\,\mathbb{I} = \log \dim \mathcal{H}_\Sigma$$

Quantum Entanglement and Topology

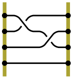
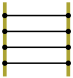
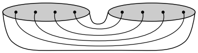
Space wiring (Preliminary classification)

The amount of entanglement is characterized by the topology:

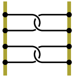
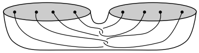
- separable states

 $\sim$  $= |0\rangle|0\rangle$

- Maximally entangled states

 or  $\sim$  $= |0\rangle|0\rangle + |1\rangle|1\rangle$

- Less entangled states (non-unitary matrices)

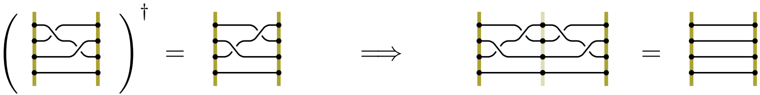
 $\sim$ 

More tangling – less entanglement!

Quantum Entanglement and Topology

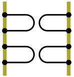
Unitarity and topology

Hermitian conjugation amounts to inverting the diagram exchanging undercrossings and overcrossings



Braiding is a natural unitary operation. However,

No natural inverse topological operation to undo the tangling. Finally,

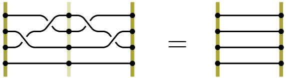
diagrams like  are projectors

Quantum Entanglement and Topology

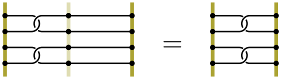
Entanglement conversion

Maximally entangled state can be converted to any state in the same SLOCC class with certainty

- Local unitaries do not affect entanglement



- Local stochastic operation can only be undone probabilistically

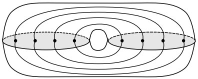


One can apply such an operation but one cannot undo such an operation

Nonmaximally entangled state can be converted to a state with higher entanglement only probabilistically

Homework Problems

- Verify that for $SU(2)$ Chern-Simons and fundamental Wilson lines

log [] $= \log 2$

- Compute the coefficient of the following states in the computation basis $|0\rangle$ and $|1\rangle$:

Verify the statements about their entanglement properties