KNOTS AND QUANTUM MECHANICS (I)

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SIMONS FOUNDATION

Main Goals

- Introduce a correspondence between Quantum Mechanics and Topological Spaces
- Introduce Topological Quantum Field Theories with elements of Knot Theory
- Demonstrate how concepts of Quantum Mechanics can be appear in the topological interpretation and how the construction can be useful beyond Quantum Entanglement

Lecture Plan

1. Introduction to knot theory
2. Topological quantum mechanics and quantum entanglement
3. Applications in quantum mechanics, quantum information and 'quantum gravity'

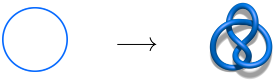
Plan of the First Lecture

- Introduce basics of knots and their invariants
- Give examples of methods of computing topological invariants

Basics of Knots

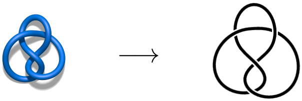
What is a **knot**?

Knot is an embedding of a circle in 3D, $\mathcal{K} : \mathbb{S}^1 \to \mathbb{R}^3$. For example,

$\mathcal{K} :$ 

Link is an embedding of multiple circles $\mathcal{L} : \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \to \mathbb{R}^3$

Knot diagram is a projection from 3D to a 2D plain, which preserves the information about the “topological order” in the crossings



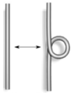
Basics of knots

Equivalence relation

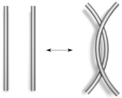
Two knots are called homeomorphic (isotopic) if they can be transformed into each other without cutting lines

Theorem [**Alexander, Briggs'26; Reidemester'27**]

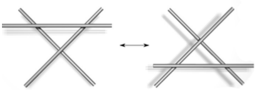
Two diagrams of the same knot can be transformed into each other (up to a planar isotopy) through the following 3 *Reidemeister moves*:



I move



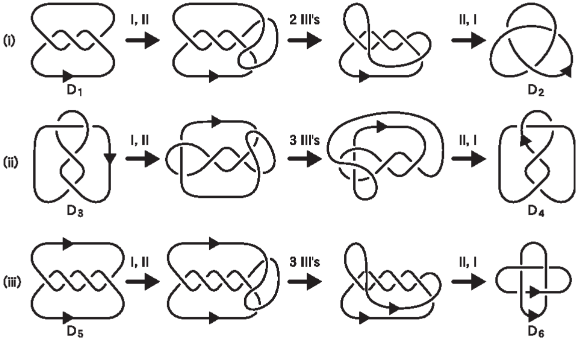
II move



III move

Basics of Knots

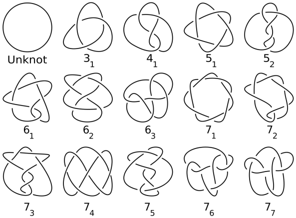
Reidemeister moves



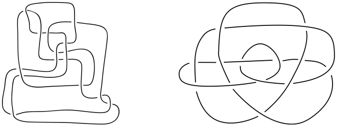
Basics of Knots

Main problem of knot theory

- How many knots are there?



- How can we distinguish two knots? [Thistlethwaite,Ochiai, `Wikipedia`]

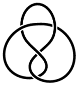


- Can we tell if a knot is trivial (unknot)?

Possible solution: construct a function of knot $i(\mathcal{L})$ that is an invariant of the topology, that is $i(\hat{\mathcal{L}}) = i(\mathcal{L})$ if $\hat{\mathcal{L}}$ and $\mathcal{L}$ are isotopic

Topological Invariants of Knots

Invariant function

i :  $\longrightarrow \quad \mathbb{C}\,,\mathbb{C}[t]\,,\mathbb{C}[t,q]$

One can hope that such a function will

- distinguish the trivial knot (unknot)
- distinguish any two knots/links that are not isotopic
- be efficiently calculable

Topological Invariants of Knots

Brief history of knot invariants

- Gauss linking number – beginning of the XIX century $i(\mathcal{L}) \in \mathbb{Z}$
- Alexander (Conway) polynomial – 1923 (1969) $i(\mathcal{L}) \in \mathbb{C}[t]$
- Jones polynomial – 1984 $i(\mathcal{L}) \in \mathbb{C}[q]$
- HOMFLY (PT) polynomial – 1985 (1987) $i(\mathcal{L}) \in \mathbb{C}[a, z]$
- Khovanov homology – 1990s
- de Floer homology – 2003

It is known that homologies can detect an unknot. It is believed that HOMFLY-PT can distinguish any pair of nonisotopic knots

Topological Invariants of Knots

Topological invariants in physics

- Gauss law: integral of the electric flux through a closed surface counts the amount of charge enclosed by the surface:

$$Q = \int_{\mathcal{S}} d^2\mathbf{n} \cdot \mathbf{E}$$

- Ampere's law: circulation of the magnetic field in a circuit counts the amount of current enclosed by the circuit:

$$\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \sum_k I_k$$

In these examples, the results do not depend on the geometry of the problem: surface $\mathcal{S}$ (circuit $\mathcal{C}$) can be arbitrarily deformed, unless it crosses charges (currents)

Topological Invariants of Knots

Gauss linking number

- Let us consider a current I running in circuit γ_1
- At point $\vec{x}$ current circuit γ_1 creates a magnetic field (Biot-Savart)

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \oint_{\gamma_1} \frac{d\vec{y} \times (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}, \qquad \vec{y} \in \gamma_1$$

- At points $\vec{x} \in \gamma_2$ use the Ampere's law:

$$\oint_{\gamma_2} \vec{B} \cdot d\vec{x} = \frac{\mu_0 I}{4\pi} \oint_{\gamma_2} \oint_{\gamma_1} \frac{(d\vec{y} \times [\vec{x}-\vec{y}]) \cdot d\vec{x}}{|\vec{x}-\vec{y}|^3} = \mu_0 I \, \delta_{\gamma_1,\gamma_2}$$

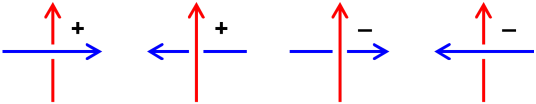
$\delta_{\gamma_1,\gamma_2} = \pm 1$ if there is a linking between the circuits (otherwise it is 0)

$$Lk \equiv \delta_{\gamma_1,\gamma_2} = \frac{1}{4\pi} \oint_{\gamma_2} \oint_{\gamma_1} \frac{(d\vec{y} \times [\vec{x}-\vec{y}]) \cdot d\vec{x}}{|\vec{x}-\vec{y}|^3} \qquad (\textit{Gauss linking number})$$

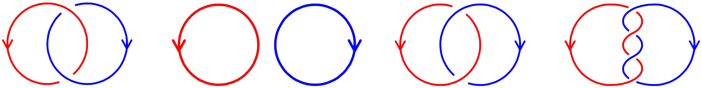
Topological Invariants of Knots

How to calculate the linking number?

- Choose directions in the loops
- Associate ± 1 to each crossing according to



- Calculate the sum of ± 1 at crossings and divide by 2

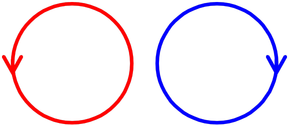


$Lk = -1$ $\quad$ $Lk = 0$ $\quad$ $Lk = 1$ $\quad$ $Lk = 2$

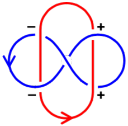
Topological Invariants of Knots

Unlinking problem

Number *Lk* has a relatively low sensitivity to the topology. For example, the same value is obtained for



and



Besides, *Lk* applies only to 2-component links

Topological Invariants of Knots

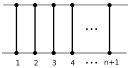
Artin's braid group

Braid group B_n is a generalization of the symmetric group S_n (permutations). It can be defined in terms of generators:

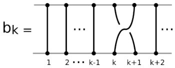
$$B_n \equiv \{b_k,\ k = 1, \dots, n-1 |\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1},\ i = 1, \dots, n-2; \\ b_i b_j = b_j b_i,\ |i-j| > 1\}$$

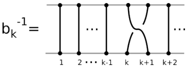
This definition has a representation on n-strand diagrams:

- identity

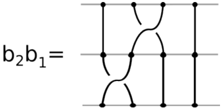


- generators and inverse





- multiplication=concatenation



Topological Invariants of Knots

Braids and knots

Theorem [**Alexander'23**] Any knot or link can be obtained as tracelike closure of some braid

We can associate the following expression to the above knot

$$\mathrm{Tr}\left[U_1^{-1}U_1^{-1}U_2U_2U_1^{-1}U_2U_1U_2^{-1}U_1^{-1}U_1^{-1}\right]$$

where U_k is some representation of the braid group generators

- One would like Tr to respect isotopies of the knot, i.e. be an invariant

Topological Invariants of Knots

Markov trace

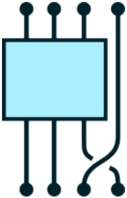
A function of the braid with the following properties produces a knot invariant:

1. (*cyclicity*) given braids α and β

$$\mathrm{Tr}\ \alpha\beta \ = \ \mathrm{Tr}\ \beta\alpha$$

2. (*Markov property*) for $\beta \in B_n$ consider $\beta \cdot b_n \in B_{n+1}$, then

$$\mathrm{Tr}_{B_n}\beta = \mathrm{Tr}_{B_{n+1}}\beta \cdot b_n$$



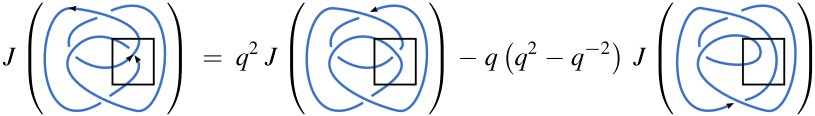
Jones (1984) realized that he knows traces with similar properties in the context of type II_1 von Neumann algebras $\longrightarrow$ Jones polynomial $J(\mathcal{K})$

Topological Invariants of Knots

Skein relations

[Conway'60s]

- Some topological invariants (Jones polynomials) satisfy linear relations



Consecutively applying the skein relations the diagram can be reduced to a linear combination of simpler diagrams, eventually to

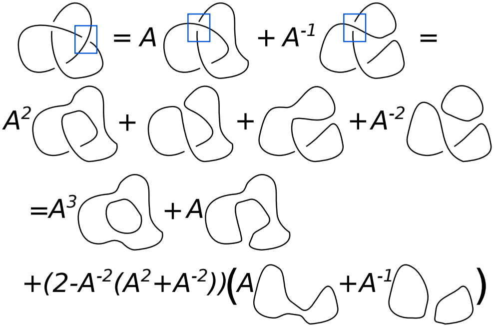
- $J(\circ) = 1$
- $J(\mathcal{K} \cup \circ) = -(q^2 + q^{-2}) J(\mathcal{K})$

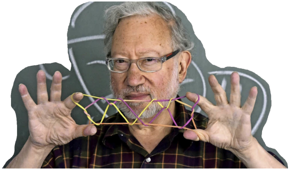
Topological Invariants of Knots

Calculus for the trefoil knot

Alternative version of the skein relations (Kauffman)

For the trefoil 3_1 $(A^4 = q)$





$$= (1+A^4)(A^3+A^{-5}-A^{-9})$$

$$= A^7(1+A^4)J_{3_1}(A^{-4})$$

Matrix representation

Braid group

Consider matrix $R : V_2 \otimes V_2 \to V_2 \otimes V_2$

$$R = \begin{pmatrix} A & & & \\ & 0 & A^{-1} & \\ & A^{-1} & A - A^{-3} & \\ & & & A \end{pmatrix}$$

The following set of matrices generate the braid group:

$$\begin{aligned} b_1 &= R \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \cdots \\ b_2 &= \mathbb{I}_2 \otimes R \otimes \mathbb{I}_2 \otimes \cdots \\ b_3 &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes R \otimes \cdots \\ &\cdots \end{aligned}$$

Here one associates a line (strand) to every copy of space V_2 (that is to $\mathbb{I}_2$)

Matrix representation

Markov trace

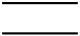
Markov trace can be defined as follows. First, define $q^H : V_2 \to V_2$:

$$q^H = \begin{pmatrix} -A^2 & \\ & -A^{-2} \end{pmatrix}$$

Then the trace

$$\mathrm{Tr}_{\mathrm{M}} X = \mathrm{Tr}\left((q^H)^{\otimes n} X \right) .$$

will respect Markov's properties and will be consistent with the following definition of the Jones (Kauffman bracket) polynomial:

- $J(\circ) = d$
- $J(\mathcal{K} \cup \circ) = d \cdot J(\mathcal{K})$
-  $= A$  $+ A^{-1}$ 

Simple self-consistency: $d = \mathrm{Tr}_{\mathrm{M}} \mathbb{I}_2 = -A^2 - A^{-2}$

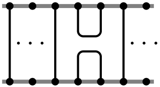
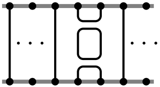
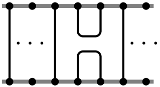
Matrix representation

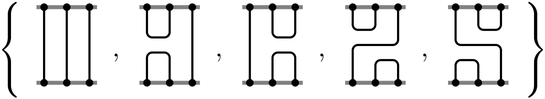
Temperley-Lieb algebra

TL algebra can be defined in terms of generators:

$$
\begin{aligned}
TL_n \equiv \{u_k,\ k = 1, \ldots, n-1 \,|\quad & u_k^2 = d \cdot u_k,\ k = 1, \ldots, n-1; \\
& u_i u_j = u_j u_i,\ |i-j| > 1; \\
& u_i u_{i+1} u_i = u_i,\ i = 1, \ldots, n-2; \\
& u_{i+1} u_i u_{i+1} = u_{i+1},\ i = 1, \ldots, n-2\}
\end{aligned}
$$

This definition also has a representation on n-strand diagrams:

$u_k =$  $\quad u_k^2 =$  $= d \cdot$ 

$TL_3 = \text{span}$ 

Matrix representation

TL and braid group

Exercise: If u_k are generators of TL_n algebra and $d = -A^2 - A^{-2}$ then

$$b_k \;=\; A\,\mathbb{I} + A^{-1} u_k$$

are generators of braid group B_n.

Recall the skein relation!

 $= A$  $+ A^{-1}$ 

Hence, we have a matrix representation for the TL algebra

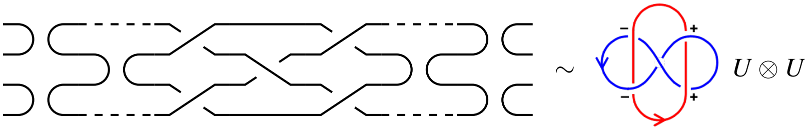
$U \;\equiv$  $= A\,R - A^2\,\mathbb{I}_4 = \begin{pmatrix} 0 & & & \\ & -A^2 & 1 & \\ & 1 & -A^{-2} & \\ & & & 0 \end{pmatrix}$

Matrix representation

Jones polynomials beyond Markov

- Trace on the TL algebra induces a trace on the braid group and vice versa

With operator U we can also compute nontracial closures of braids:

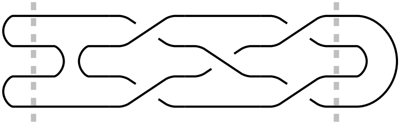


This will be the basis of our quantum mechanics construction

Matrix representation

Jones polynomials as matrix elements

It is interesting to be able to cast closures as matrix elements



$$= \langle \Psi | B | \Phi \rangle$$

Think of U as

$U \equiv$  $= |u\rangle\langle u|\,,\qquad$ for some $\qquad |u\rangle \in V_2 \otimes V_2$

Possible solution:

$$|u\rangle \;=\; \pm(0, iA, -iA^{-1}, 0)$$

To be precise, this would give correct U if $|u\rangle \equiv \langle u|$. No conjugation!

Matrix representation

Pseudounitary representation

- Let us assume that $A = e^{i\theta}$. In particular, $A^* = A^{-1}$

Note that $R^\dagger \neq R^{-1}$

- Introduce

$$\Sigma = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

Then

$$(\Sigma \otimes \Sigma) R^\dagger (\Sigma \otimes \Sigma)^\dagger = R^{-1}$$

Σ is a metric on V_2. This conjugation is equivalent to real transposition of vectors: $\langle u| \equiv (|u\rangle)^\dagger \Sigma = (|u\rangle)^T$

$|u\rangle =$  , $\langle u| =$  , $|u\rangle \otimes |u\rangle =$  , $\ldots$

Knots and Quantum Field Theory

Chern-Simons theory

3D theory with gauge group $SU(N)$:

$$S = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \, \mathrm{Tr} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right)$$

- Equations of motion tell that the physical fields are trivial:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$$

  – field configurations are pure gauges

- CS theory is *topological* because S does not depend on metric. As a result, it has vanishing energy-momentum tensor

$$T_{\mu\nu} \propto \frac{\delta S}{\delta g^{\mu\nu}} = 0$$

Knots and Quantum Field Theory

Particles

Among the gauge transformations of CS some are singular. They slightly modify equations of motion:

$$F^a_{\mu\nu} \propto T^a \epsilon_{\mu\nu\rho} \frac{dx^\rho}{ds} \delta(\vec{x} - \vec{x}(s))$$

This can be viewed as coupling of gauge fields to curves (particle worldlines)

The theory has some nontrivial observables: expectation values of Wilson loops

- Wilson loop operator (for a given closed contour $\mathcal{L}$)

$$W(\mathcal{L}, R) = \mathrm{Tr}_R \, P \exp \left(\oint_{\mathcal{L}} A_\mu dx^\mu \right)$$

- Its expectation value is computated in the standard fashion:

$$\langle W(\mathcal{L}, R) \rangle = \frac{\int \mathcal{D}A_\mu \, W(\mathcal{L}, R) e^{iS[A]}}{\int \mathcal{D}A_\mu \, e^{iS[A]}}$$

Knots and Quantum Field Theory

Jones polynomials

Witten (1988): expectation values of Wilson loop operators in Chern-Simons with group $SU(2)$ calculate Jones polynomials:

$$J_{\mathcal{L},R}(q) \ \sim \ \langle W(\mathcal{L},R)\rangle_{k,SU(2)} \ , \qquad q \ = \ e^{\frac{2\pi i}{k+2}} \quad (A = q^{1/4})$$

- the original Jones polynomials correspond to the choice of the fundamental representation
- for arbitrary R one obtains a generalization known as *colored Jones polynomials*
- for $SU(N)$ the polynomials are HOMFLY-PT

Knots in 3D Manifolds

Quantization

It is the simplest to quantize CS theory on $\Sigma \times S^1$, for Riemann surface Σ

- In the Coulomb gauge $A_0 = 0$ the action is quadratic:

$$S \sim \int d^3x \, A_1^a \dot{A}_2^a$$

- There is a Gauss constraint

$$F_{12}^a = 0$$

What are the available phase space configurations satisfying the constraint?

- The answer depends on the topology of Σ

Knots in 3D Manifolds

Sphere and other topologies

Monodromies (traces of the Wilson loops) are the most generic gauge invariant observables

- On the sphere S^2 there are no nontrivial monodromies: the quantization leads to a trivial Hilbert space

$$\dim \mathcal{H}_{S^2} \;=\; 1$$

- On the torus T^2 the nontrivial degrees of freedom correspond monodromies along the noncontractible cycles
- Integer coupling constant k limits the number of possible windings. For $SU(2)$

$$\dim \mathcal{H}_{T^2} \;=\; k+1$$

- For arbitrary genus

$$\dim \mathcal{H}_{\mathcal{M}_g} \;=\; (k+2)^{g-1} \sum_{j=0}^{k} \sin\left(\frac{\pi j}{k+2}\right)^{2-2g}$$

Knots in 3D Manifolds

Adding particles to S^2

Nondynamical particles are charges sourcing the gauge fields:

$$F^a_{\mu\nu} \propto T^a \epsilon_{\mu\nu\rho} \frac{dx^\rho}{ds} \delta(\vec{x} - \vec{x}(s))$$

- The nontrivial monodromies can wind around the charges (charge≡representation)
- On a closed manifold the consistency of the monodromies requires the total charge (total spin) to vanish
- The dimension of the Hilbert space corresponds to the number of the ways the representations can form a singlet

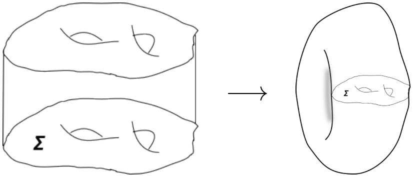
$$R_1 \otimes R_2 \otimes \cdots \ni R_\emptyset, \qquad \text{fusion algebra:} \quad \Phi_i \otimes \Phi_j = \oplus_k N^k_{ij} \Phi_k$$

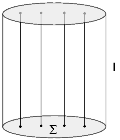
Number of charges	0	1	2	3	4
$\dim \mathcal{H}_{S^2}$	1	$\delta^R_\emptyset$	$\delta^{R_2}_{R_1}$	$N^{R_3}_{R_1 R_2}$	$\cdots$

Knots in 3D Manifolds

Simple partition functions

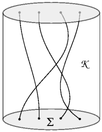
Compute CS path integral on $\Sigma \times S^1$





- Since the Hamiltonian $H = 0$, the path integral looks like a computation of a trace of the identity operator on $\mathcal{H}_\Sigma$

$$Z(\Sigma \times S^1) \; = \; \mathrm{Tr}_{\,\mathcal{H}_\Sigma} \mathbb{I} \; = \; \dim \mathcal{H}_\Sigma$$



$$Z = \mathrm{Tr}_{\,\mathcal{H}_\Sigma} \mathcal{K}$$

This also applies to S^2 with charges

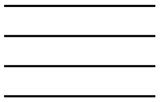
For an evolution generated by a map $\mathcal{K}$, Z computes its trace

Knots in 3D Manifolds

In the matrix presentation

- Consider $SU(2)$ theory with charges $j = 1/2$ irreps
- The naive calculation of the trace

$$\mathrm{Tr}\left(\quad \right) = 16$$



- Markov trace Tr_M computes the invariant in S^3 and yields d^4

Can we get the invariant in $Z(S^2 \times S^1)$?

- In the next lecture we will construct appropriate projectors P_2 (or $P_{\mathcal{H}_\Sigma}$ in general), so that

$$\mathrm{Tr}\, P_{\mathcal{H}_\Sigma} \mathbb{I} = \dim \mathcal{H}_\Sigma \qquad \text{and} \qquad \mathrm{Tr}\, P_{\mathcal{H}_\Sigma} \mathcal{K} \quad \text{– invariant on } S^2 \times S^1$$

Homework Exercises

1. If u_k are generators of TL_n algebra and $d = -A^2 - A^{-2}$ then

$$b_k \;=\; A\,\mathbb{I} + A^{-1} u_k$$

are generators of braid group B_n.

2. Calculate the Jones polynomial of the figure-eight knot using skein relations in Kauffman's conventions

3. Calculate the same invariant using matrix representation as the Markov trace of a braid

4. Calculate the same invariant using matrix representation as a matrix element of a braid